

# Quantum electrodynamics of relativistic bound states with cutoffs.

**Jean-Marie Barbaroux\***

*Centre de Physique Théorique, Campus de Luminy  
Case 907, 13288 Marseille Cedex 9, France*

**Mouez Dimassi†**

*CNRS-UMR 7539, Département de Mathématiques  
Institut Galilée, Université Paris-Nord  
93430 Villetaneuse, France*

**Jean-Claude Guillot‡**

*CNRS-UMR 7539, Département de Mathématiques  
Institut Galilée, Université Paris-Nord  
93430 Villetaneuse, France*

*and*

*CNRS-UMR 7641, Centre de Mathématiques Appliquées, Ecole Polytechnique  
91128 Palaiseau Cedex, France*

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## Abstract

We consider an Hamiltonian with ultraviolet and infrared cutoffs, describing the interaction of relativistic electrons and positrons in the Coulomb potential with photons in Coulomb gauge. The interaction includes both interaction of the current density with transversal photons and the Coulomb interaction of charge density with itself.

We prove that the Hamiltonian is self-adjoint and has a ground state for sufficiently small coupling constants.

\*E-mail: barbarou@cpt.univ-mrs.fr

†E-mail: dimassi@math.univ-paris13.fr

‡E-mail: guillot@cmapx.polytechnique.fr

# 1 Introduction

In Refs. [17] an Hamiltonian with cutoffs describing relativistic electrons and positrons in a Coulomb potential interacting with transversal photons in Coulomb gauge is considered. In that note [17], results concerning the self-adjointness of the Hamiltonian and the existence of a ground state for it are announced.

In this article, we consider the full model of QED by adding the Coulomb interaction of the charge density with itself to the Hamiltonian described in [17]. We thus study an Hamiltonian with ultraviolet and infrared cutoff functions with respect to the momenta of photons, but also with respect to the momenta of electrons and positrons. The total Hamiltonian in the Fock space of electrons, positrons and photons is then well defined in the Furry picture.

In this paper, we prove results concerning the self-adjointness of the Hamiltonian and the existence of a ground state when the coupling constants are sufficiently small.

In [9], Bach, Fröhlich and Sigal proved the existence of a ground state for the Pauli-Fierz Hamiltonian with an ultraviolet cutoff for photons, and for sufficiently small values of the fine structure constant, without introducing an infrared cutoff. Their result has been extended by Griesemer, Lieb and Loss [22] under the binding condition, and finally by Barbaroux, Chen and Vugalter [11], and Lieb and Loss [31]. For related results see [4, 5, 6, 19, 39].

No-pair Hamiltonians for relativistic electrons in QED have been recently considered in [10, 18, 30, 34]. In [3], Arai has analyzed the hamiltonian of a Dirac particle interacting with the quantum radiation field. In [38], an Hamiltonian without conservation of the particle number is studied. In [1], the scattering theory for the spin fermion model is studied.

The case of relativistic electrons in classical magnetic fields was studied earlier in [33] and [23]. There, it was proven instability for the Brown and Ravenhall model in the free picture. In [23] it is even deduced from this result that instability also holds in QED context, i.e. for the Brown and Ravenhall model in the free picture, coupled to the quantized radiation field with or without cutoff.

In our case, working in the Furry picture and imposing both electronic and photonic cutoffs prevent from instability.

Our methods follow those of [2, 9, 7, 8, 20], in which the spectral theory of the spin-boson and Pauli-Fierz Hamiltonians is studied.

The infrared conditions on the cutoff functions with respect to the momenta of photons are stronger than those appearing in [9] and [22].

Some of the results obtained in this paper have been announced in [17] and [12]. The case where we are only dealing with the Wick ordering corrected Coulomb interaction of charge density with itself can be found in [13].

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## 2 The quantization of the Dirac-Coulomb field

The Dirac electron of mass  $m_0$  in a Coulomb potential is described by the following Hamiltonian:

$$H_D := H_D(e) = c\boldsymbol{\alpha} \cdot \frac{1}{i}\nabla + \beta m_0 c^2 - \frac{e^2 Z}{|x|},$$

acting in the Hilbert space  $L^2(\mathbb{R}^3; \mathbb{C}^4)$  with domain  $\mathfrak{D}(H_D) = H^1(\mathbb{R}^3; \mathbb{C}^4)$ , the Sobolev space of order 1. Here  $\hbar = 1$ . We refer to [40] for a discussion of the Dirac operator (see also [36]).

Here  $e$  is the negative charge of the electron,  $-Ze$  is the positive charge of the nucleus and  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ ,  $\beta$  are the Dirac matrices in the standard form:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

where  $\sigma_i$  are the usual Pauli matrices.

It is well known that  $H_D(e)$  is self-adjoint for  $Z \leq 118$  ([40, Theorem 4.4]). The eigenstates of the Dirac-Coulomb operator  $H_D$  are labelled by the angular momentum quantum numbers  $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ ,  $m_j = -j, -j+1, \dots, j-1, j$ , by the spin-orbit quantum number  $k_j = \pm(j + \frac{1}{2}) = \pm 1, \pm 2, \dots$  and by the quantum number  $n = 0, 1, 2, \dots$ , counting the non-degenerate eigenvalues of the Dirac radial operator associated with  $k_j$ . From now on we set  $\gamma = (j, m_j, k_j)$ .

Let  $\psi_{\gamma,n}$  denote the eigenstates of the Hamiltonian  $H_D$ . We have

$$H_D(e)\psi_{\gamma,n} = E_{\gamma,n}\psi_{\gamma,n},$$

with

$$E_{\gamma,n} = m_0 c^2 \left( 1 + \frac{(Ze^2)^2/c^2}{(n + \sqrt{k_j^2 - (Ze^2)^2/c^2})^2} \right)^{-1/2}$$

Each eigenvalue is degenerate. The infimum of the discrete spectrum is an isolated eigenvalue of multiplicity two. Together with the bound state energy levels the continuum energy levels are given by

$$\pm\omega(p), \quad \omega(p) = (m_0^2 c^4 + p^2)^{\frac{1}{2}}, \quad p = |\mathbf{p}| \tag{1}$$

Here  $\mathbf{p}$  is the momentum of the electron.

Let  $\psi_{\gamma,\pm}(p, x)$  denote the continuum eigenstates of  $H_D(e)$ . We then have

$$H_D \psi_{\gamma,\pm}(p, x) = \pm \omega(p) \psi_{\gamma,\pm}(p, x).$$

Here,  $\psi_{\gamma,n}$  and  $\psi_{\gamma,\pm}(p, x)$  are normalized in such a way that

$$\begin{aligned} \int_{\mathbb{R}^3} \psi_{\gamma,n}^\dagger(x) \psi_{\gamma',n'}(x) d^3x &= \delta_{nn'} \delta_{\gamma\gamma'}, \\ \int_{\mathbb{R}^3} \psi_{\gamma,\pm}^\dagger(p, x) \psi_{\gamma',\pm}(p', x) d^3x &= \delta_{\gamma\gamma'} \delta(p - p'), \\ \int_{\mathbb{R}^3} \psi_{\gamma,\pm}^\dagger(p, x) \psi_{\gamma',\mp}(p', x) d^3x &= 0, \\ \int_{\mathbb{R}^3} \psi_{\gamma,n}^\dagger(x) \psi_{\gamma',\pm}(p, x) d^3x &= 0. \end{aligned}$$

Here  $\psi_{\gamma,\pm}^\dagger(p, x)$  (resp.  $\psi_{\gamma,n}^\dagger(x)$ ) is the adjoint spinor of  $\psi_{\gamma,\pm}(p, x)$  (resp.  $\psi_{\gamma,n}(x)$ ). The spinors  $(\psi_{\gamma,n})_{\gamma,n}$  and  $(\psi_{\gamma,\pm})_\gamma$  generate a spectral representation of  $H_D(e)$ .

According to the hole theory [14, 27, 28, 36, 37, 40, 41], the absence in the Dirac theory of an electron with energy  $E < 0$  and charge  $e$  is equivalent to the presence of a positron with energy  $-E > 0$  and charge  $-e$ .

Let us split the Hilbert space  $\mathfrak{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$  into

$$\mathfrak{H}_{c^-} = P_{(-\infty, 0]}(H_D)\mathfrak{H}, \quad \mathfrak{H}_d = P_{(0, m_0 c^2)}(H_D)\mathfrak{H}, \quad \mathfrak{H}_{c^+} = P_{[m_0 c^2, +\infty)}(H_D)\mathfrak{H}.$$

Here  $P_I(H_D)$  denotes the spectral projection of  $H_D$  corresponding to the interval  $I$ . Since  $(\psi_{\gamma,n})_{\gamma,n}$  is a basis in  $\mathfrak{H}_d$ , every  $\psi \in \mathfrak{H}_d$  can be written as

$$\psi(x) = \text{L.i.m } \sum_{\gamma,n} b_{\gamma,n} \psi_{n,\gamma}(x),$$

with

$$\sum_{\gamma,n} |b_{\gamma,n}|^2 < \infty.$$

Hence we can identify  $\mathfrak{H}_d$  with the Hilbert space  $\tilde{\mathfrak{H}}_d = \bigoplus_\gamma F_\gamma$ , where  $F_\gamma$  is a closed subspace of  $L^2(\mathbb{R}_+)$ . Precisely, there exists a unitary operator  $U_d$  from  $\mathfrak{H}_d$  onto  $\tilde{\mathfrak{H}}_d$  given by

$$U_d \psi = \left( \sum_n b_{\gamma,n} \mathbb{1}_{[n, n+1)}(p) \right)_\gamma, \quad \psi \in \mathfrak{H}_d$$

where  $p \in \mathbb{R}_+$ .

Similarly,  $\mathfrak{H}_{c\pm}$  can be identified with  $\tilde{\mathfrak{H}}_{c\pm} := \oplus_\gamma L^2(\mathbb{R}_+)$  by using the unitary operators  $U_{c\pm}$  :

$$U_{c\pm}\psi = \left( \text{L.i.m } \int \psi_{\gamma,\pm}^\dagger(p, x) \psi(x) dx \right)_\gamma,$$

where  $\psi \in \mathfrak{H}_{c\pm}$ .

## 2.1 The Fock space for electrons and positrons

Let  $\mathfrak{G}$  be any separable Hilbert space. Let  $\otimes_a^n \mathfrak{G}$  denotes the antisymmetric  $n$ -th tensor power of  $\mathfrak{G}$  appropriate to Fermi-Dirac statistics. We define the Fermi-Fock space over  $\mathfrak{G}$ , denoted by  $\mathfrak{F}_a(\mathfrak{G})$ , to be the direct sum

$$\mathfrak{F}_a(\mathfrak{G}) = \bigoplus_{n=0}^{\infty} \otimes_a^n \mathfrak{G},$$

where  $\otimes_a^0 \mathfrak{G} := \mathbb{C}$ . The state  $\Omega$  will denote the vacuum vector, i.e., the vector  $(1, 0, 0, \dots)$ .

Let  $\mathfrak{F}_{a,d}$ ,  $\mathfrak{F}_{a,+}$  and  $\mathfrak{F}_{a,-}$  be the Fermi-Fock spaces over  $\tilde{\mathfrak{H}}_d$ ,  $\tilde{\mathfrak{H}}_{c+}$  and  $\tilde{\mathfrak{H}}_{c-}$  respectively.  $\Omega_d$ ,  $\Omega_+$  and  $\Omega_-$  denote the associated vacua.

In the Furry picture, the Fermi-Fock space for electrons and positrons, denoted by  $\mathfrak{F}_D$ , is then the following Hilbert space

$$\mathfrak{F}_D = \mathfrak{F}_{a,d} \otimes \mathfrak{F}_{a,+} \otimes \mathfrak{F}_{a,-} \quad (2)$$

The vector  $\Omega_d \otimes \Omega_+ \otimes \Omega_-$  is the vacuum of electrons and positrons.

One has

$$\mathfrak{F}_D = \bigoplus_{q,r,s=0}^{\infty} \mathfrak{F}_a^{(q,r,s)}$$

where  $\mathfrak{F}_d^{(q,r,s)} = (\otimes_a^q \tilde{\mathfrak{H}}_d) \otimes (\otimes_a^r \tilde{\mathfrak{H}}_{c+}) \otimes (\otimes_a^s \tilde{\mathfrak{H}}_{c-})$ .

Let us remark that  $\mathfrak{F}_D$  is unitarily equivalent to  $\mathfrak{F}_a(\tilde{\mathfrak{H}}_d \oplus \tilde{\mathfrak{H}}_{c+} \oplus \tilde{\mathfrak{H}}_{c-})$ . (See [15] and [16]).

## 2.2 Creation and annihilation operators

For every  $\varphi \in \mathfrak{H}$  we define in  $\mathfrak{F}_a(\mathfrak{H})$  the annihilation operator, denoted by  $b(\varphi)$ , which maps  $\otimes_a^{n+1} \mathfrak{H}$  into  $\otimes_a^n \mathfrak{H}$  :

$$\begin{aligned} b(\varphi) (A_{n+1}(\varphi_1 \otimes \dots \otimes \varphi_{n+1})) \\ = \frac{\sqrt{n+1}}{(n+1)!} \sum_{\sigma} sgn(\sigma) (\varphi, \varphi_{\sigma(1)}) \varphi_{\sigma(2)} \otimes \dots \otimes \varphi_{\sigma(n+1)} \end{aligned}$$

where  $\varphi_i \in \mathfrak{H}$ .

The creation operator, denoted by  $b^*(\varphi)$ , is the adjoint of  $b(\varphi)$ . The operators  $b^*(\varphi)$  and  $b(\varphi)$  are bounded in  $\mathfrak{F}_a(\mathfrak{H})$  and  $\|b(\varphi)\| = \|b^*(\varphi)\| = \|\varphi\|$ .

We now define the annihilation and creation operators in  $\mathfrak{F}_D$ .

### 2.2.1 Bound states

For every  $(\gamma, n)$ , we define in  $\mathfrak{F}_D$  the annihilation operator, denoted by  $b_{\gamma,n}$ , which maps  $\mathfrak{F}^{(q+1,r,s)}$  into  $\mathfrak{F}^{(q,r,s)}$  :

$$\begin{aligned} & b_{\gamma,n} (A_{q+1}(f_1 \otimes \dots \otimes f_{q+1}) \otimes A_r(g_1 \otimes \dots \otimes g_r) \otimes A_s(h_1 \otimes \dots \otimes h_s)) \\ &= [b(U_d \psi_{\gamma,n}) A_{q+1}(f_1 \otimes \dots \otimes f_{r+1})] \otimes A_r(g_1 \otimes \dots \otimes g_r) \otimes A_s(h_1 \otimes \dots \otimes h_s) \end{aligned}$$

where  $f_i \in \tilde{\mathfrak{H}}_d$ ,  $g_j \in \tilde{\mathfrak{H}}_{c+}$  and  $h_k \in \tilde{\mathfrak{H}}_{c-}$ .

The creation operator  $b_{\gamma,n}^*$  is the adjoint of  $b_{\gamma,n}$ .  
 $b_{\gamma,n}^*$  and  $b_{\gamma,n}$  are bounded operators in  $\mathfrak{F}_D$ .

### 2.2.2 Electrons in the continuum

For every  $g \in \tilde{\mathfrak{H}}_{c+}$ , we define in  $\mathfrak{F}_D$  the annihilation operator, denoted by  $b_+(g)$ , which maps  $\mathfrak{F}^{(q,r+1,s)}$  into  $\mathfrak{F}^{(q,r,s)}$  as follows

$$\begin{aligned} & b_+(g) (A_q(f_1 \otimes \dots \otimes f_q) \otimes A_{r+1}(g_1 \otimes \dots \otimes g_{r+1}) \otimes A_s(h_1 \otimes \dots \otimes h_s)) \\ &= A_q(f_1 \otimes \dots \otimes f_q) \otimes [(-1)^q b(g) A_{r+1}(g_1 \otimes \dots \otimes g_{r+1})] \otimes A_s(h_1 \otimes \dots \otimes h_s) \end{aligned}$$

The creation operator  $b_+^*(g)$  is the adjoint of  $b_+(g)$ .  $b_+^*(g)$  and  $b_+(g)$  are bounded operators in  $\mathfrak{F}_D$ .

We set, for every  $\psi \in \tilde{\mathfrak{H}}_{c+}$ ,

$$\begin{aligned} b_{\gamma,+}(\psi) &= b_+(P_\gamma^+ \psi) \\ b_{\gamma,+}^*(\psi) &= b_+^*(P_\gamma^+ \psi) \end{aligned}$$

where  $P_\gamma^+$  is the projection of  $\tilde{\mathfrak{H}}_{c+}$  onto the  $\gamma$ -th component.

### 2.2.3 Positrons

For every  $h \in \tilde{\mathfrak{H}}_{c-}$ , we define in  $\mathfrak{F}_D$  the annihilation operator, denoted by  $b_-(h)$ , which maps  $\mathfrak{F}^{(q,r,s+1)}$  into  $\mathfrak{F}^{(q,r,s)}$  :

$$\begin{aligned} & b_-(h)(A_q(f_1 \otimes \dots \otimes f_q) \otimes A_r(g_1 \otimes \dots \otimes g_r) \otimes A_{s+1}(h_1 \otimes \dots \otimes h_{s+1})) \\ &= A_q(h_1 \otimes \dots \otimes f_q) \otimes A_r(g_1 \otimes \dots \otimes g_r) \otimes [(-1)^{q+r} b(h) A_{s+1}(h_1 \otimes \dots \otimes h_{s+1})] \end{aligned}$$

The creation operator  $b_-^*(h)$  is the adjoint of  $b_-(h)$ .  $b_-^*(h)$  and  $b_-(h)$  are bounded operators in  $\mathfrak{F}_D$ .

We set for every  $\psi \in \tilde{\mathfrak{H}}_{c-}$ ,

$$b_{\gamma,-}(\psi) = b_-(P_\gamma^- \psi)$$

$$b_{\gamma,-}^*(\psi) = b_-^*(P_\gamma^- \psi)$$

where  $P_\gamma^-$  is the projection of  $\tilde{\mathfrak{H}}_{c-}$  onto the  $\gamma$ -th component.

A simple computation shows that the following anti-commutation relations hold

$$\{b_{\gamma,n}^*, b_{\beta,m}\} = \delta_{\gamma,\beta} \delta_{n,m}$$

$$\{b_{\gamma,n}^*, b_{\beta,m}^*\} = 0 = \{b_{\gamma,n}, b_{\beta,m}\}$$

$$\{b_{\gamma,+}(\psi_1), b_{\beta,+}^*(\psi_2)\} = \delta_{\gamma,\beta} (P_\gamma^+ \psi_1, P_\gamma^+ \psi_2)_{L^2(\mathbb{R}_+)}, \quad \psi_i \in \tilde{\mathfrak{H}}_{c+}$$

$$\{b_{\gamma,-}(\psi_1), b_{\beta,-}^*(\psi_2)\} = \delta_{\gamma,\beta} (P_\gamma^- \psi_1, P_\gamma^- \psi_2)_{L^2(\mathbb{R}_+)}, \quad \psi_i \in \tilde{\mathfrak{H}}_{c-}$$

Furthermore

$$\{b_{\gamma,\pm}^\#(\psi), b_{\beta,n}^\#\} = 0.$$

Here  $b^\#$  is  $b$  or  $b^*$ .

$$\{b_{\gamma,+}(\psi_1), b_{\beta,-}(\psi_2)\} = \{b_{\gamma,+}(\psi_1), b_{\beta,-}^*(\psi_2)\} = 0,$$

$$\{b_{\gamma,+}^*(\psi_1), b_{\beta,-}(\psi_2)\} = \{b_{\gamma,+}^*(\psi_1), b_{\beta,-}^*(\psi_2)\} = 0,$$

where  $\psi_1 \in \tilde{\mathfrak{H}}_{c+}$  and  $\psi_2 \in \tilde{\mathfrak{H}}_{c-}$ . One should remark that, in contrast to [40], the charge conjugation operator is not included in the definition of the annihilation and creation operators for the positrons.

Our definition is close to the hole theory and is the one occurring in many text books in Quantum Field Theory as in [27], [28], [37] and [41]. The other method for the quantization of the Dirac field is close to the so-called symmetric theory of charge. Both approaches are described in [28].

As in [35, chapter X], we introduce operator-valued distributions  $b_{\gamma,\pm}(p)$  and  $b_{\gamma,\pm}^*(p)$  such that we write

$$b_{\gamma,\pm}(\psi) = \int_{\mathbb{R}^+} dp \, b_{\gamma,\pm}(p) \overline{(P_\gamma^\pm \psi)(p)}$$

$$b_{\gamma,\pm}^*(\psi) = \int_{\mathbb{R}^+} dp \, b_{\gamma,\pm}^*(p) (P_\gamma^\pm \psi)(p)$$

where  $\psi \in \tilde{\mathfrak{H}}_{c+} \oplus \tilde{\mathfrak{H}}_{c-}$ .

We now give a representation of  $b_{\gamma,\pm}(p)$  and  $b_{\gamma,\pm}^*(p)$ . Let  $\mathfrak{D}_D$  denote the set of smooth vectors  $\Phi \in \mathfrak{F}_D$  for which  $\Phi^{(q,r,s)}$  has a compact support and  $\Phi^{(q,r,s)} = 0$  for all but finitely many  $(q, r, s)$ .

For every  $(p, \gamma)$ ,  $b_{\gamma,+}(p)$  maps  $\mathfrak{F}_a^{(q,r+1,s)} \cap \mathfrak{D}_D$  into  $\mathfrak{F}_a^{(q,r,s)} \cap \mathfrak{D}_D$  and we have

$$(b_{\gamma,+}(p)\Phi)^{(q,r,s)}(p_1, \gamma_1, \dots, p_q, \gamma_q; p_1, \gamma_1, \dots, p_r, \gamma_r; p_1, \gamma_1, \dots, p_s, \gamma_s) = \sqrt{r+1}(-1)^q \Phi^{(q,r+1,s)}(p_1, \gamma_1, \dots, p_q, \gamma_q; p, \gamma, p_1, \gamma_1, \dots, p_r, \gamma_r; p_1, \gamma_1, \dots, p_s, \gamma_s)$$

$b_{\gamma,+}^*(p)$  is then given by:

$$\begin{aligned} & (b_{\gamma,+}^*(p)\Phi)^{(q,r+1,s)}(p_1, \gamma_1, \dots, p_q, \gamma_q; p_1, \gamma_1, \dots, p_{r+1}, \gamma_{r+1}; p_1, \gamma_1, \dots, p_s, \gamma_s) = \\ & \frac{(-1)^q}{\sqrt{r+1}} \sum_{i=1}^{r+1} (-1)^{i+1} \delta_{\gamma_i \gamma} \delta(p - p_i) \\ & \Phi^{(q,r,s)}(p_1, \gamma_1, \dots, p_q, \gamma_q; p_1, \gamma_1, \dots, \widehat{p_i, \gamma_i}, \dots, p_{r+1}, \gamma_{r+1}; p_1, \gamma_1, \dots, p_s, \gamma_s) \end{aligned}$$

where  $\widehat{\cdot}$  denotes that the  $i$ -th variable has to be omitted.

Similarly  $b_{\gamma,-}(p)$  maps  $\mathfrak{F}_a^{(q,r,s+1)} \cap \mathfrak{D}_D$  into  $\mathfrak{F}_a^{(q,r,s)} \cap \mathfrak{D}_D$  such that

$$\begin{aligned} & (b_{\gamma,-}(p)\Phi)^{(q,r,s)}(p_1, \gamma_1, \dots, p_q, \gamma_q; p_1, \gamma_1, \dots, p_r, \gamma_r; p_1, \gamma_1, \dots, p_s, \gamma_s) = \\ & \sqrt{s+1}(-1)^{q+r} \Phi^{(q,r,s+1)}(p_1, \gamma_1, \dots, p_q, \gamma_q; p_1, \gamma_1, \dots, p_r, \gamma_r; p, \gamma, p_1, \gamma_1, \dots, p_s, \gamma_s) \end{aligned}$$

$b_{\gamma,-}^*(p)$  is then given by

$$\begin{aligned} & (b_{\gamma,-}^*(p)\Phi)^{(q,r,s+1)}(p_1, \gamma_1, \dots, p_q, \gamma_q; p_1, \gamma_1, \dots, p_r, \gamma_r; p_1, \gamma_1, \dots, p_{s+1}, \gamma_{s+1}) = \\ & \frac{1}{\sqrt{s+1}} (-1)^{q+r} \sum_{i=1}^{s+1} (-1)^{i+1} \delta_{\gamma_i \gamma} \delta(p - p_i) \\ & \Phi^{(q,r,s)}(p_1, \gamma_1, \dots, p_q, \gamma_q; p_1, \gamma_1, \dots, p_r, \gamma_r; p_1, \gamma_1, \dots, \widehat{p_i, \gamma_i}, \dots, p_{s+1}, \gamma_{s+1}) \end{aligned}$$

Let us recall that  $\Phi^{(q,r,s)}$  is antisymmetric in the bound states, the electron in the continuum and the positron variables separately. We have

$$\{b_{\gamma,+}(p), b_{\gamma',+}^*(p')\} = \{b_{\gamma,-}(p), b_{\gamma',-}^*(p')\} = \delta_{\gamma,\gamma'} \delta(p - p')$$

Any other anti-commutation relation is equal to zero. In particular we have

$$\{b_{\gamma,\pm}(p), b_{\gamma',n}^\#(p)\} = \{b_{\gamma,\pm}^*(p), b_{\gamma',n}^\#(p)\} = 0$$

where  $b^\#$  is  $b$  or  $b^*$ .

#### 2.2.4 The Hamiltonian for the quantized Dirac-Coulomb field

The quantization of the Dirac-Coulomb Hamiltonian  $H_D$ , denoted by  $d\Gamma(H_D)$ , is now given by

$$\begin{aligned} d\Gamma(H_D) &= \sum_{\gamma,n} E_{\gamma,n} b_{\gamma,n}^* b_{\gamma,n} + \sum_{\gamma} \int dp \omega(p) b_{\gamma,+}^*(p) b_{\gamma,+}(p) \\ &\quad + \sum_{\gamma} \int dp \omega(p) b_{\gamma,-}^*(p) b_{\gamma,-}(p), \end{aligned}$$

with  $\omega(p)$  given in (1). The operator  $d\Gamma(H_D)$  is the Hamiltonian of the quantized Dirac-Coulomb field. It is well defined on the dense subset  $\mathfrak{D}_D$  and it is essentially self-adjoint on  $\mathfrak{D}_D$ . The self-adjoint extension will be also denoted by  $d\Gamma(H_D)$  with domain  $\mathfrak{D}(d\Gamma(H_D))$ . Moreover the operator number of electrons and positrons, denoted by  $N_D$ , is given by

$$N_D = \sum_{\gamma,n} b_{\gamma,n}^* b_{\gamma,n} + \sum_{\gamma} \int dp b_{\gamma,+}^*(p) b_{\gamma,+}(p) + \sum_{\gamma} \int dp b_{\gamma,-}^*(p) b_{\gamma,-}(p).$$

The operator  $N_D$  is essentially self-adjoint on  $\mathfrak{D}_D$ . The self-adjoint extension will be also denoted by  $N_D$  with domain  $\mathfrak{D}(N_D)$ .

Let  $\Delta$  denote the set of all eigenvalues of the operator

$$d\Gamma(H_D)_d := \sum_{\gamma,n} E_{\gamma,n} b_{\gamma,n}^* b_{\gamma,n}$$

Each  $E_{\gamma,n}$  is in  $\Delta$  and we have

$$0 < E_0 = \inf_{\gamma,n} E_{\gamma,n}, \quad \text{and} \quad \inf_{E \in \Delta} E = 0.$$

0 is the eigenvalue associated with  $\Omega_D$ . Furthermore

$$\sigma(d\Gamma(H_D)) = \Delta \cup [m_0 c^2, \infty).$$

The set  $[m_0 c^2, \infty)$  is the absolutely continuous spectrum of  $d\Gamma(H_D)$ . Let us remark that, for  $E \in \Delta$  with  $E > m_0 c^2$ ,  $E$  is an eigenvalue embedded in the continuous spectrum.

### 2.3 The Fock space for transversal photons

The one photon Hilbert space is given by  $L^2(\mathbb{R}^3, \mathbb{C}^2)$ . For every  $f$  in  $L^2(\mathbb{R}^3, \mathbb{C}^2)$ , we shall write  $f(k, \mu)$  where  $k \in \mathbb{R}^3$  is the momentum variable of the photon and  $\mu =$

$\mu = 1, 2$  is its polarization index associated with two given independent real transversal polarizations  $\varepsilon_\mu(k)$  of the photon in the Coulomb gauge such that  $\varepsilon_\mu(k) \cdot \varepsilon_{\mu'}(k) = \delta_{\mu\mu'}$  and  $\varepsilon_\mu(k) \cdot k = 0$ .

Let  $\mathfrak{F}_{ph}$  denote the Fock space for transversal photons :

$$\mathfrak{F}_{ph} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^n}$$

where  $L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^0} = \mathbb{C}$ . Here  $L^2(\mathbb{R}^3, \mathbb{C}^2)^{\otimes_s^n}$  is the symmetric  $n$ -tensor power of  $L^2(\mathbb{R}^3, \mathbb{C}^2)$  appropriate for Bose-Einstein statistics.

For  $f \in L^2(\mathbb{R}^3, \mathbb{C}^2)$ , the annihilation and creation operators, denoted by  $a(f)$  and  $a^*(f)$  respectively, are now given by:

$$a(f) = \sum_{\mu=1,2} a_\mu(f(\cdot, \mu))$$

$$a^*(f) = \sum_{\mu=1,2} a_\mu^*(f(\cdot, \mu))$$

where

$$(a_\mu(f(\cdot, \mu))\Psi)^{(n)}(k_1, \mu_1, k_2, \mu_2, \dots, k_n, \mu_n) = \sqrt{n+1} \int d^3k \overline{f(k, \mu)} \Psi^{(n+1)}(k, \mu, k_1, \mu_1, k_2, \mu_2, \dots, k_n, \mu_n)$$

$$(a_\mu^*(f(\cdot, \mu))\Psi)^{(n)}(k_1, \mu_1, \dots, k_n, \mu_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^n f(k_i, \mu_i) \widehat{\Psi^{(n-1)}}(k_1, \mu_1, \dots, \widehat{k_i}, \widehat{\mu_i}, \dots, k_n, \mu_n)$$

where  $\widehat{\cdot}$  denotes that the  $i$ -th variable has to be omitted. Note that  $a_\mu^*(f(\cdot, \mu))$  and  $a_\mu(f(\cdot, \mu))$  are linear and anti-linear with respect to  $f$  respectively, so that we can introduce operator valued distributions, i.e., fields  $a_\mu(k)$  and  $a_\mu^*(k)$  such that

$$a_\mu(f(\cdot, \mu)) = \int d^3k \overline{f(k, \mu)} a_\mu(k)$$

and

$$a_\mu^*(f(\cdot, \mu)) = \int d^3k f(k, \mu) a_\mu^*(k)$$

where  $f(\cdot, \mu) \in L^2(\mathbb{R}^3)$ .

Let  $\mathfrak{D}_{ph}$  denote the set of smooth vectors  $\Psi \in \mathfrak{F}_{ph}$  for which  $\Psi^{(n)}$  has a compact support and  $\Psi^{(n)} = 0$  for all but finitely many  $n$ . Then, for any  $\Psi \in \mathfrak{D}_{ph}$ , the action of  $a_\mu(k)$  and  $a_\mu^*(k)$  is given by

$$\begin{aligned} & (a_\mu(k)\Psi)^{(n)}(k_1, \mu_1, k_2, \mu_2, \dots, k_n, \mu_n) \\ &= \sqrt{n+1}\Psi^{(n+1)}(k, \mu, k_1, \mu_1, \dots, k_n, \mu_n) \\ & (a_\mu^*(k)\Psi)^{(n+1)}(k_1, \mu_1; k_2, \mu_2, \dots, k_{n+1}, \mu_{n+1}) \\ &= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \delta_{\mu_i, \mu} \delta(k_i - k) \Psi^{(n)}(k_1, \mu_1, \dots, \widehat{k_i, \mu_i}, \dots, k_{n+1}, \mu_{n+1}) \end{aligned}$$

We have the canonical commutation relations

$$[a_\mu(k), a_{\mu'}^*(k')] = \delta_{\mu\mu'} \delta(k - k')$$

Any other commutation relation is equal to zero.

The Hamiltonian of the quantized electromagnetic field, denoted by  $H_{ph}$ , is

$$H_{ph} = \sum_{\mu=1,2} \int d^3k \omega(k) a_\mu^*(k) a_\mu(k)$$

where  $\omega(k) = c|k|$ .  $H_{ph}$  is essentially self-adjoint on  $\mathfrak{D}_{ph}$ .

The state  $(1, 0, \dots) \in \mathfrak{F}_{ph}$  is the vacuum of photons and will be denoted  $\Omega_{ph}$ .

The spectrum of  $H_{ph}$  consists of an absolutely continuous spectrum covering  $[0, +\infty)$  and a simple eigenvalue, equal to zero, whose the corresponding eigenvector is the vacuum state  $\Omega \in \mathfrak{F}_{ph}$ .

## 2.4 The total Hamiltonian and the main results

The Fock space for electrons, positrons and photons is the following Hilbert space:

$$\mathfrak{F} = \mathfrak{F}_D \otimes \mathfrak{F}_{ph}.$$

$\Omega = \Omega_D \otimes \Omega_{ph}$  is the vacuum of  $\mathfrak{F}$ .

The free Hamiltonian for electrons, positrons and photons, denoted by  $H_0$ , is the following operator in  $\mathfrak{F}$ :

$$H_0 = d\Gamma(H_D) \otimes \mathbb{1}_{ph} + \mathbb{1}_D \otimes H_{ph}$$

$H_0$  is a self-adjoint operator with domain

$$\mathfrak{D}(d\Gamma(H_D) \otimes \mathbb{1}_{ph}) \cap \mathfrak{D}(\mathbb{1}_D \otimes H_{ph}).$$

The operator  $H_0$  has the same point spectrum as  $d\Gamma(H_D)$  and its continuous spectrum covers the half axis  $[0, \infty)$ . Hence the point spectrum of  $d\Gamma(H_D)$  is embedded in the continuous spectrum of  $H_0$ . The eigenfunctions of  $H_0$  corresponding to the eigenvalues  $E$  have the form  $\varphi \otimes \Omega_{ph}$ , where  $\varphi$  are the eigenfunctions of  $d\Gamma(H_D)$  corresponding to the eigenvalues  $E$ .

Let us now describe the interaction between electrons, positrons and photons in Coulomb gauge that we consider.

Let us first recall the physical interaction in QED with the Coulomb gauge.

The quantized Dirac-Coulomb field is given by

$$\begin{aligned} \psi(x) &= \sum_{\gamma,n} b_{\gamma,n} \psi_{\gamma,n}(x) \\ &+ \sum_{\gamma} \int_{\mathbb{R}^+} dp b_{\gamma,+}(p) \psi_{\gamma,+}(p, x) + \sum_{\gamma} \int_{\mathbb{R}^+} dp b_{\gamma,-}^*(p) \psi_{\gamma,-}(p, x) \end{aligned}$$

together with

$$\begin{aligned} \psi^\dagger(x) &= \sum_{\gamma,n} b_{\gamma,n}^* \psi_{\gamma,n}^\dagger(x) + \sum_{\gamma} \int_{\mathbb{R}^+} dp b_{\gamma,+}^*(p) \psi_{\gamma,+}^\dagger(p, x) \\ &+ \sum_{\gamma} \int_{\mathbb{R}^+} dp b_{\gamma,-}(p) \psi_{\gamma,-}^\dagger(p, x) \end{aligned}$$

for the quantized adjoint spinor.

Here  $b_{\gamma,n}$ ,  $b_{\gamma,n}^*$ ,  $b_{\gamma,\pm}(p)$  and  $b_{\gamma,\pm}^*(p)$  are the annihilation and creation operators defined above.

The density of charge is then given by

$$\rho(x) = e : \psi^\dagger(x) \psi(x) :$$

and the density of current by

$$j(x) = ec : \psi^\dagger(x) \alpha \psi(x) :$$

Here  $::$  is the normal ordering and  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ .

The interaction of electrons and positrons with photons is given by two terms. The first one is

$$\frac{e^2}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3x d^3x' \frac{\rho(x) \rho(x')}{|x - x'|}, \quad (3)$$

and describes the Coulomb interaction of densities of charges. The second one,

$$-e \int_{\mathbb{R}^3} d^3x \ j(x) \cdot A(x),$$

describes the interaction between the current and the transversal photons. Here  $A(x)$  is the quantized electromagnetic field in Coulomb gauge

$$A(x) = \sum_{\mu=1,2} \left( \frac{\hbar}{2\varepsilon_0(2\pi)^3} \right)^{1/2} \int \frac{d^3k}{\sqrt{2\omega_{ph}(k)}} (\varepsilon_\mu(k) e^{ik \cdot x} a_\mu(k) + \varepsilon_\mu(k) e^{-ik \cdot x} a_\mu^*(k))$$

(See [14] and [21]),  $a_\mu^*(k)$  and  $a_\mu(k)$  are the creation and annihilation operators on  $\mathfrak{F}_{ph}$ .

Then, at a formal level, the interaction terms can be expressed in terms of the annihilation and creation operators  $b_{\gamma,n}$ ,  $b_{\gamma,n}^*$ ,  $b_{\gamma,\pm}(p)$ ,  $b_{\gamma,\pm}^*(p)$ ,  $a_\mu(k)$  and  $a_\mu^*(k)$ . Note that for the interaction between the current and the transversal photons, the products of the  $b$ 's and  $b^*$ 's must be Wick normal ordered. Furthermore, the Coulomb interaction of densities of charges (3) as it stands, is not Wick normal ordered. For both physical and mathematical reasons (see [35] and [41]) it is more convenient to rewrite it as a sum of Wick normal ordered terms by using the anti-commutation relations.

It is a known fact that we have to introduce several cutoff functions in the Dirac-Coulomb field and the electromagnetic vector potential in order to get a well defined total Hamiltonian in the Fock space (see [32]).

Thus the interaction between the electrons, positrons and photons consists of two terms. The first term in the interaction, denoted by  $H_I^{(1)}$ , is given by

$$\begin{aligned} H_I^{(1)} &= \\ &\sum_{\gamma,\gamma',n,\ell} \sum_{\mu=1,2} \int d^3k \left( G_{d,\gamma,\gamma',n,\ell}^\mu(k) b_{\gamma,n}^* b_{\gamma',\ell} a_\mu^*(k) + \text{h.c.} \right) \\ &+ \sum_{\epsilon=+,-} \sum_{\gamma,\gamma',n} \sum_{\mu=1,2} \int d^3k dp \left( G_{d,\epsilon,\gamma,\gamma',n}^\mu(p; k) \left( b_{\gamma,n}^* b_{\gamma',\epsilon}(p) \right. \right. \\ &\quad \left. \left. + b_{\gamma,\epsilon}^*(p) b_{\gamma',n} \right) a_\mu^*(k) + \text{h.c.} \right) \\ &+ \sum_{\gamma,\gamma'} \sum_{\mu=1,2} \int d^3k dp dp' \left( G_{+,-,\gamma,\gamma'}^\mu(p, p'; k) \left( b_{\gamma,+}^*(p) b_{\gamma',-}^*(p') \right. \right. \\ &\quad \left. \left. + b_{\gamma,-}(p) b_{\gamma',+}(p') \right) a_\mu^*(k) + \text{h.c.} \right) \\ &+ \sum_{\epsilon=+,-} \sum_{\gamma,\gamma'} \sum_{\mu=1,2} \int d^3k dp dp' \left( G_{\epsilon,\epsilon,\gamma,\gamma'}^\mu(p, p'; k) b_{\gamma,\epsilon}^*(p) b_{\gamma',\epsilon}(p') a_\mu^*(k) + \text{h.c.} \right) \end{aligned}$$

For the second term let us introduce some notations. In the case of electrons,  $\xi$  will be equal to  $(\gamma, p)$  and  $(\gamma, n)$  with  $\int d\xi = \sum_\gamma \int dp + \sum_{\gamma,n}$ . In the case of positrons,  $\xi$  will be equal to  $(\gamma, p)$  with  $\int d\xi = \sum_\gamma \int dp$  and the  $L^2$  norm of any function of  $\xi$  will be the sum of the  $L^2$  norm with respect to the continuous part of the measure and the  $L^2$  norm with respect to the discrete part of the measure. The second term of the interaction, denoted by  $H_I^{(2)}$ , is an operator in  $\mathfrak{F}_D$  given by

$$\begin{aligned}
H_I^{(2)} = & \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4) b_+^*(\xi_1) b_-^*(\xi_2) b_+(\xi_3) b_-(\xi_4) \\
& + \sum_{\epsilon=+, -} \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F_\epsilon^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4) b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_\epsilon(\xi_3) b_\epsilon(\xi_4) \\
& + \sum_{\substack{\epsilon, \epsilon' = +, - \\ \epsilon \neq \epsilon'}} \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \left( F_{\epsilon, \epsilon'}^{(3)}(\xi_1, \xi_2, \xi_3, \xi_4) b_\epsilon^*(\xi_1) b_\epsilon(\xi_2) b_{\epsilon'}(\xi_3) b_\epsilon(\xi_4) \right. \\
& \quad \left. - \overline{F_{\epsilon, \epsilon'}^{(3)}(\xi_4, \xi_2, \xi_3, \xi_1)} b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_{\epsilon'}^*(\xi_3) b_\epsilon(\xi_4) \right) \\
& + \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \left( F^{(4)}(\xi_1, \xi_2, \xi_3, \xi_4) b_+(\xi_1) b_+(\xi_2) b_-(\xi_3) b_-(\xi_4) \right. \\
& \quad \left. + \overline{F^{(4)}(\xi_4, \xi_2, \xi_3, \xi_1)} b_+^*(\xi_1) b_+^*(\xi_2) b_-^*(\xi_3) b_-^*(\xi_4) \right) \\
& + \sum_{\epsilon=\pm} \int d\xi_1 d\xi_2 F_\epsilon^{(5)}(\xi_1, \xi_2) b_\epsilon^*(\xi_1) b_\epsilon(\xi_2) \\
& + \sum_{\substack{\epsilon, \epsilon' = +, - \\ \epsilon \neq \epsilon'}} \int d\xi_1 d\xi_2 \left( F_{\epsilon, \epsilon'}^{(6)}(\xi_1, \xi_2) b_\epsilon^*(\xi_1) b_{\epsilon'}^*(\xi_2) + \overline{F_{\epsilon, \epsilon'}^{(6)}(\xi_2, \xi_1)} b_{\epsilon'}(\xi_1) b_\epsilon(\xi_2) \right)
\end{aligned}$$

where, for  $\xi = (\gamma, n)$ ,  $b^\sharp(\xi) := b_{\gamma, n}^\sharp$ . Furthermore, we suppose that

$$\begin{aligned}
F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4) &= \overline{F^{(1)}(\xi_3, \xi_4, \xi_1, \xi_2)} \\
F_\epsilon^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4) &= \overline{F_\epsilon^{(2)}(\xi_4, \xi_3, \xi_2, \xi_1)}
\end{aligned}$$

and

$$F_\epsilon^{(5)}(\xi_1, \xi_2) = \overline{F_\epsilon^{(5)}(\xi_2, \xi_1)}, \quad \epsilon = +, -$$

**Definition 2.1.** *The Hamiltonian for relativistic electrons and positrons in a Coulomb potential interacting with photons in Coulomb gauge that we consider is given by*

$$H(g_1, g_2) := H_0 + g_1 H_I^{(1)} + g_2 H_I^{(2)} \otimes \mathbb{1},$$

where  $g_i$ ,  $i = 1, 2$ , are real coupling constants.

It is easy to show that  $H(g)$  is a symmetric operator in  $\mathcal{F}$  as soon as the kernels  $F^{(i)}$ 's and  $G^{(i)}$ 's are square integrable. With stronger conditions on the  $F^{(i)}$ 's and  $G^{(i)}$ 's, we recover a self-adjoint operator with a ground state, as stated in Theorem 2.2 and 2.3 below.

Let, for  $\beta = 0, 1, 2$ ,

$$\begin{aligned} C_\beta &= \sum_{\mu=1,2} \left( \sum_{\gamma,\gamma',n,\ell} \int_{\mathbb{R}^3} |G_{d,\gamma,\gamma',n,\ell}^\mu(k)|^2 \omega(k)^{-\beta} d^3k \right)^{1/2} \\ &\quad + \sum_{\varepsilon=+,-} \sum_{\mu=1,2} \left( \sum_{\gamma,\gamma',n} \int_{\mathbb{R}^3 \times \mathbb{R}^+} |G_{d,\varepsilon,\gamma,\gamma',n}^\mu(p;k)|^2 \omega(k)^{-\beta} dp d^3k \right)^{1/2} \\ &\quad + \sum_r \sum_{\mu=1,2} \left( \sum_{\gamma,\gamma'} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} |G_{r,\gamma,\gamma'}^\mu(p,p';k)|^2 \omega(k)^{-\beta} dp dp' d^3k \right)^{1/2} \end{aligned}$$

where  $r = \{+,+\}, \{+,-\}, \{-,-\}$ . We also set

$$F_\epsilon^{(2),a}(\xi_1, \xi_2, \xi_3, \xi_4) := F_\epsilon^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4) - F_\epsilon^{(2)}(\xi_1, \xi_2, \xi_4, \xi_3)$$

We now state our main results.

**Theorem 2.2.** *We assume that every  $F^{(j)} \in L^2$  ( $j = 1, 2, 3, 4, 5, 6$ ). Furthermore, we suppose that  $C_0 < \infty$ ,  $C_1 < \infty$  and*

$$\frac{|g_1|}{\sqrt{E_0}} C_1 + \frac{|g_2|}{E_0} \left( \frac{1}{\sqrt{2}} \|F^{(1)}\| + \|F_+^{(2),a}\| + \|F_-^{(2),a}\| \right) < 1 \quad (4)$$

*Then,  $H(g_1, g_2)$  is self-adjoint on the domain  $\mathfrak{D}(H_0)$ .*

**Theorem 2.3.** *We assume that every  $F^{(j)} \in L^2$  ( $j = 1, 2, 3, 4, 5, 6$ ), and that  $C_0 < \infty$ ,  $C_1 < \infty$  and  $C_2 < \infty$ . Furthermore, we suppose that (4) holds true. Then there exists  $g_0 > 0$  such that for  $|g_1| + |g_2| \leq g_0$ , the self-adjoint operator  $H(g_1, g_2)$  has a ground state.*

### 3 Proofs of Theorems 2.2 and 2.3

Let

$$\mathcal{D} = \{\gamma = (j, m_j, k_j); j = \frac{1}{2}, \frac{3}{2}, \dots, m_j = -j, \dots, j, k_j = \pm 1, \pm 2, \dots, \}$$

$$\mathcal{D}_d = \{(\gamma, n); \gamma \in \mathcal{D} \text{ and } n = 0, 1, 2, \dots\}$$

We set

$$\begin{aligned}\Sigma_+ &= \Sigma_- = \mathbb{R}^+ \times \mathcal{D} \\ \Sigma_d &= \mathbb{R}^+ \times \mathcal{D}_d\end{aligned}$$

From now on, we will write  $\xi = (p, \gamma)$  where  $(p, \gamma)$  belongs to  $\Sigma_+$  and  $\Sigma_-$  respectively and  $\xi_d = (p, \gamma, n)$  where  $(p, \gamma, n)$  belongs to  $\Sigma_d$ . Then

$$L^2(\Sigma_\pm) = \left\{ f : \Sigma_\pm \rightarrow \mathbb{C}; \int_{\Sigma_\pm} |f(\xi)|^2 d\xi := \sum_{\gamma \in \mathcal{D}} \int_{\mathbb{R}^+} |f(p, \gamma)|^2 dp < \infty \right\}.$$

Thus the map

$$(f_\gamma(\cdot))_{\gamma \in \mathcal{D}} \in \bigoplus_{\gamma \in \mathcal{D}} L^2(\mathbb{R}^+) \longrightarrow f(\xi) = f(p, \gamma) \in L^2(\Sigma_\pm),$$

is unitary and we will identify the space  $\tilde{\mathfrak{H}}_{c+}$  (resp.  $\tilde{\mathfrak{H}}_{c-}$ ) introduced in Section 2 with  $L^2(\Sigma_+)$  (resp.  $L^2(\Sigma_-)$ ).

Similarly we identify the space  $\tilde{\mathfrak{H}}_d$  with a closed subspace, denoted by  $F$ , of  $L^2(\Sigma_d)$ . According to Section 2,  $F$  is the closure of the following subspace

$$\left\{ \sum_{(\gamma, n) \in \mathcal{D}_d} c_{\gamma, n} \mathbb{1}_{[n, n+1]}(p); \sum_{(\gamma, n) \in \mathcal{D}_d} |c_{\gamma, n}|^2 < \infty \right\}$$

Hence the Fermi-Fock space  $\mathfrak{F}_D$  will be identified with

$$\mathfrak{F}_a(F) \otimes \mathfrak{F}_a(L^2(\Sigma_+)) \otimes \mathfrak{F}_a(L^2(\Sigma_-))$$

Let us now define the new annihilation and creation operators. Recall that (See [40])

$$\begin{aligned}\left\| \int_{\mathbb{R}^+} dp \overline{h_\pm(p, \gamma)} b_{\gamma \pm}(p) \right\| &= \left\| \int dp h_\pm(p, \gamma) b_{\gamma, \pm}^*(p) \right\| \\ &= \|h_\pm(\cdot, \gamma)\|_{L^2(\mathbb{R}^+)}\end{aligned}$$

Thus the series  $\sum_{\gamma \in \mathcal{D}} \int_{\mathbb{R}^+} dp \overline{h_\pm(p, \gamma)} b_{\gamma, \pm}(p)$  and  $\sum_{\gamma \in \mathcal{D}} \int_{\mathbb{R}^+} dp h_\pm(p, \gamma) b_{\gamma, \pm}^*(p)$  are normally convergent in  $\ell^2(\mathcal{D})$ ,  $B(\mathfrak{F}_a(\tilde{\mathfrak{H}}_{c\pm}))$  respectively, for every  $h_\pm$  in  $L^2(\Sigma_\pm)$ .

Here  $B(\mathfrak{F}_a(\tilde{\mathfrak{H}}_{c\pm}))$  is the set of bounded operators in  $\mathfrak{F}_a(\tilde{\mathfrak{H}}_{c\pm})$  respectively.

We then set, for every  $h_{\pm}$  in  $L^2(\Sigma_{\pm})$  respectively,

$$\begin{aligned} b_{\pm}(h_{\pm}) &:= \int d\xi \overline{h_{\pm}(\xi)} b_{\pm}(\xi) = \sum_{\gamma \in \mathcal{D}} \int_{\mathbb{R}^+} dp \overline{h_{\pm}(p, \gamma)} b_{\gamma, \pm}(p) \\ b_{\pm}^*(h_{\pm}) &:= \int d\xi h_{\pm}(\xi) b_{\pm}^*(\xi) = \sum_{\gamma \in \mathcal{D}} \int_{\mathbb{R}^+} dp h_{\pm}(p, \gamma) b_{\gamma, \pm}^*(p) \end{aligned}$$

where  $b_{\gamma, \pm}(p)$  and  $b_{\gamma, \pm}^*(p)$  are defined in Section 2.

Similarly the series  $\sum_{(\gamma, n) \in \mathcal{D}_d} \overline{c_{\gamma, n}} b_{\gamma, n}$  and  $\sum_{(\gamma, n) \in \mathcal{D}_d} c_{\gamma, n} b_{\gamma, n}^*$ , where  $\sum_{(\gamma, n) \in \mathcal{D}_d} |c_{\gamma, n}|^2 < \infty$ , are normally convergent in  $\ell^2(\mathcal{D}_d, B(\mathfrak{F}_a(\tilde{\mathfrak{H}}_d)))$ .

Thus, for any  $h_d(\xi_d) = (c_{\gamma, n} \mathbf{1}_{[n, n+1]}(p))_{(\gamma, n) \in \mathcal{D}_d} \subset L^2(\Sigma_d)$ , we set

$$b_d(h_d) = \sum_{(\gamma, n) \in \mathcal{D}_d} \overline{c_{\gamma, n}} b_{\gamma, n},$$

$$b_d^*(h_d) = \sum_{(\gamma, n) \in \mathcal{D}_d} c_{\gamma, n} b_{\gamma, n}^*,$$

where  $b_{\gamma, n}$  and  $b_{\gamma, n}^*$  are defined in Section 2.2.1.

The new canonical anti-commutation relations are the following ones

$$\{b_{\pm}(h_{\pm}^1), b_{\pm}^*(h_{\pm}^2)\} = \langle h_{\pm}^1, h_{\pm}^2 \rangle_{L^2(\Sigma_{\pm})} \quad (5)$$

$$\{b_{\pm}^{\#}(h_{\pm}^1), b_{\pm}^{\#}(h_{\pm}^2)\} = 0 \quad (6)$$

$$\{b_{\pm}^{\#}(h_{\pm}), b_d^{\#}(h_d)\} = 0 \quad (7)$$

$$\{b_d(h_d^1), b_d^*(h_d^2)\} = (h_d^1, h_d^2)_{L^2(\Sigma_d)} \quad (8)$$

Here  $b^{\#}$  is  $b$  or  $b^*$ .

We now have the following lemmas whose proofs are well-known.

**Lemma 3.1.** *Let  $f$  be in  $L^2(\mathbb{R}^3, \mathbb{C}^2; \omega(k)^{-\beta} d^3 k)$  for  $\beta = 0, 1$ . We have*

$$\begin{aligned} \|a(f)\Psi\|_{\mathfrak{F}_{ph}}^2 &\leq \left( \sum_{\mu=1,2} \int \frac{|f(k, \mu)|^2}{\omega(k)} d^3 k \right) \|H_{ph}^{1/2}\Psi\|_{\mathfrak{F}_{ph}}^2 \\ \|a^*(f)\Psi\|_{\mathfrak{F}_{ph}}^2 &\leq \left( \sum_{\mu=1,2} \int \frac{|f(k, \mu)|^2}{\omega(k)} d^3 k \right) \|H_{ph}^{1/2}\Psi\|_{\mathfrak{F}_{ph}}^2 \\ &+ \left( \sum_{\mu=1,2} \int |f(k, \mu)|^2 d^3 k \right) \|\Psi\|_{\mathfrak{F}_{ph}}^2 \end{aligned}$$

for every  $\Psi \in \mathfrak{D}(H_{ph})$  (See [7]).

**Lemma 3.2.** For  $h_\pm \in L^2(\Sigma_\pm)$ ,  $b_\pm(h_\pm)$  and  $b_\pm^*(h_\pm)$  are bounded operators and we have

$$\|b_\pm(h)\Psi\|^2 + \|b_\pm^*(h)\Psi\|^2 = \|\Psi\|^2 \|h\|_{L^2(\Sigma_\pm)}^2.$$

In particular,

$$\|b_\pm(h)\| = \|b_\pm^*(h)\| = \|h\|_{L^2(\Sigma_\pm)}$$

for every  $\Psi \in \mathfrak{F}_a(L^2(\Sigma_\pm))$  respectively (See [40]).

**Lemma 3.3.** For every  $h_d \in F$ ,  $b_d^\#(h_d)$  is a bounded operator and we have

$$\|b_d(h_d)\Psi\|^2 + \|b_d^*(h_d)\Psi\|^2 = \|h_d\|_{L^2(\Sigma_d)}^2 \|\Psi\|^2$$

for all  $\Psi \in \mathfrak{F}_a(L^2(\Sigma_d))$ .

Let  $\{f_i, i = 1, 2, \dots\}$  (resp.  $\{g_j, j = 1, 2, \dots\}$ ,  $\{h_k, k = 1, 2, \dots\}$ ) be an orthonormal basis of  $L^2(F)$  (resp.  $L^2(\Sigma_+)$ ,  $L^2(\Sigma_-)$ ). We suppose that the  $g_j$ 's and the  $h_k$ 's are smooth functions in the Schwartz space with respect to  $p$ .

We will now consider vectors in  $\mathfrak{F}_D$  of the following form:

$$\Phi^{(\ell,m,n)} := b_d^*(f_{i_1}) \dots b_d^*(f_{i_\ell}) b_+^*(g_{j_1}) \dots b_+^*(g_{j_m}) b_-^*(h_{k_1}) \dots b_-^*(h_{k_n}) \Omega, \quad (9)$$

where  $l, m, n$  are positive integers and  $\Omega = \Omega_d \otimes \Omega_{c+} \otimes \Omega_{c-}$ . The indexes will be assumed ordered such that  $i_1 < \dots < i_l$ ,  $j_1 < \dots < j_m$  and  $k_1 < \dots < k_n$ . It is known that the set  $\{\Phi^{(\ell,m,n)}; \ell, m, n = 0, 1, 2, \dots\}$  is an orthonormal basis of  $\mathfrak{F}_D$  (See [40]). The set

$$\mathfrak{F}_{0,D} = \{\Psi \in \mathfrak{F}_D; \Psi \text{ is a finite linear combination of basis vectors of the form (9)}\}$$

is dense in  $\mathfrak{F}_D$ .

In the following propositions we investigate several operators in  $\mathfrak{F}_D$  built from product of creation operators or annihilation operators only. In the case of electrons we restrict ourselves to the electrons in the continuous spectrum. The case of electrons in the discrete spectrum is easier to deal with.

For  $G \in L^2(\Sigma_\pm \times \Sigma_\pm)$  the formal operators

$$\int_{\Sigma_\pm \times \Sigma_\pm} d\xi_1 d\xi_2 \overline{G(\xi_1, \xi_2)} b_\pm(\xi_1) b_\pm(\xi_2)$$

are defined as quadratic forms on  $\mathfrak{F}_{0,D} \times \mathfrak{F}_{0,D}$ :

$$\int_{\Sigma_\pm \times \Sigma_\pm} d\xi_1 d\xi_2 \langle \Psi, \overline{G(\xi_1, \xi_2)} b_\pm(\xi_1) b_\pm(\xi_2) \Phi \rangle,$$

where  $\Psi, \Phi \in \mathfrak{F}_{0,D}$ . Mimicking the proof of [35, Theorem X.44], we get two operators, denoted by  $A_{\pm}$ , associated with these two forms such that  $A_{\pm}$  are the unique operators in  $\mathfrak{F}_D$  so that  $\mathfrak{F}_{0,D} \subset \mathfrak{D}(A_{\pm})$  is a core for  $A_{\pm}$  and

$$A_{\pm} = \int_{\Sigma_{\pm} \times \Sigma_{\pm}} d\xi_1 d\xi_2 \overline{G(\xi_1, \xi_2)} b_{\pm}(\xi_1) b_{\pm}(\xi_2)$$

as quadratic forms on  $\mathfrak{F}_{0,D} \times \mathfrak{F}_{0,D}$ .

From now on we denote  $A_{\epsilon}$  the operators  $A_{\pm}$  where  $\epsilon = +, -$ , associated with the kernels  $G_{\epsilon}(\xi_1, \xi_2)$ . In the same way, we define as quadratic forms on  $\mathfrak{F}_{0,D} \times \mathfrak{F}_{0,D}$  the operators  $A_{\epsilon\epsilon'}$ ,  $A_{\epsilon\epsilon'\epsilon}$  and  $A_{\epsilon\epsilon'\epsilon'\epsilon'}$ , where  $\epsilon, \epsilon' = +, -$  and  $\epsilon \neq \epsilon'$  as follows

$$\begin{aligned} A_{\epsilon\epsilon'} &= \int_{\Sigma_{\epsilon} \times \Sigma_{\epsilon'}} d\xi_1 d\xi_2 \overline{G_{\epsilon\epsilon'}(\xi_1, \xi_2)} b_{\epsilon}(\xi_1) b_{\epsilon'}(\xi_2), \\ A_{\epsilon\epsilon'\epsilon} &= \int_{\Sigma_{\epsilon} \times \Sigma_{\epsilon'} \times \Sigma_{\epsilon}} d\xi_1 d\xi_2 d\xi_3 \overline{G_{\epsilon\epsilon'\epsilon}(\xi_1, \xi_2, \xi_3)} b_{\epsilon}(\xi_1) b_{\epsilon'}(\xi_2) b_{\epsilon}(\xi_3), \\ A_{\epsilon\epsilon'\epsilon'\epsilon'} &= \int_{\Sigma_{\epsilon} \times \Sigma_{\epsilon} \times \Sigma_{\epsilon'} \times \Sigma_{\epsilon'}} d\xi_1 d\xi_2 d\xi_3 d\xi_4 \overline{G_{\epsilon\epsilon'\epsilon'\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4)} b_{\epsilon}(\xi_1) b_{\epsilon}(\xi_2) b_{\epsilon'}(\xi_3) b_{\epsilon'}(\xi_4), \end{aligned}$$

with  $G_{\epsilon\epsilon'}$ ,  $G_{\epsilon\epsilon'\epsilon}$  and  $G_{\epsilon\epsilon'\epsilon'\epsilon'}$  in  $L^2$ .

Let  $\Phi^{(\ell, m, n)}$  be a vector of the form (9). In addition, for simplicity, assume that  $\{i_1, \dots, i_{\ell}\} = \{1, \dots, \ell\}$ ,  $\{j_1, \dots, j_m\} = \{1, \dots, m\}$  and  $\{k_1, \dots, k_n\} = \{1, \dots, n\}$ , i.e.

$$\Phi^{(\ell, m, n)} = \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^m b_+^*(g_j) \otimes \prod_{k=1}^n b_-^*(h_k) \Omega.$$

We claim that

$$A_+ \Phi^{(\ell, m, n)} = \sum_{1 \leq \alpha < \beta \leq m} (-1)^{\alpha+\beta} \langle G_+^a, g_{\alpha} \otimes g_{\beta} \rangle \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha, j \neq \beta}} b_+^*(g_j) \otimes \prod_{k=1}^n b_+^*(h_k) \quad (10)$$

where

$$G_+^a(\xi_1, \xi_2) = G_+(\xi_1, \xi_2) - G_+(\xi_2, \xi_1)$$

and

$$\langle G_+^a, g_{\alpha} \otimes g_{\beta} \rangle = \int \overline{G_+^a(\xi_1, \xi_2)} g_{\alpha}(\xi_1) g_{\beta}(\xi_2) d\xi_1 d\xi_2.$$

Indeed, the canonical anti-commutation relations (CAR) (5)-(8) yield

$$\begin{aligned} & \int d\xi_2 \overline{G_+(\xi_1, \xi_2)} b_+(\xi_2) \Phi^{(\ell, m, n)} \\ &= (-1)^{\ell} \sum_{\alpha=1}^m (-1)^{\alpha+1} \langle G_+(\xi_1, .), g_{\alpha} \rangle \prod_{i=1}^{\ell} b_d^*(h_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha}} b_+^*(g_j) \otimes \prod_{k=1}^n b_-^*(h_k) \Omega \quad (11) \end{aligned}$$

We have the identity  $\langle G_+(\xi_1, .), g_\alpha \rangle = \sum_{\beta=1}^{\infty} \langle G_+, g_\beta \otimes g_\alpha \rangle \overline{g_\beta(\xi_1)}$  where  $\langle G_+, g_\beta \otimes g_\alpha \rangle = \int_{\Sigma_+ \times \Sigma_+} \overline{G_+(\xi_1, \xi_2)} g_\beta(\xi_1) g_\alpha(\xi_2) d\xi_1 d\xi_2$ .

Applying  $b_+(\xi_1)$  to the right hand side of (11), using the CAR as above and integrating with respect to  $\xi_1$  we get

$$\begin{aligned} A_+ \Phi^{(\ell, m, n)} &= \sum_{\alpha=1}^{m-1} (-1)^{\alpha+1} \sum_{1 \leq \alpha < \beta \leq m} \langle G_+, g_\alpha \otimes g_\beta \rangle \\ &\quad (-1)^{\beta+1} \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha, j \neq \beta}} b_+^*(g_j) \otimes \prod_{k=1}^n b_+^*(h_k) \Omega \\ &\quad + \sum_{\alpha=1}^{m-1} (-1)^{\alpha+1} \sum_{1 \leq \alpha < \beta \leq m} \langle G_+, g_\beta \otimes g_\alpha \rangle \\ &\quad (-1)^\beta \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha, j \neq \beta}} b_+^*(g_j) \otimes \prod_{k=1}^n b_-^*(h_k) \Omega, \end{aligned} \tag{12}$$

from which we deduce (10). Similarly, we have

$$\begin{aligned} A_- \Phi^{(\ell, m, n)} &= \sum_{1 \leq \gamma < \delta \leq n} (-1)^{\gamma+\delta} \langle G_-^a, h_\gamma \otimes h_\delta \rangle \\ &\quad \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^m b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n\} \\ k \neq \gamma, k \neq \delta}} b_+^*(h_k) \Omega, \end{aligned} \tag{13}$$

where

$$G_-^a(\xi_1, \xi_2) = G_-(\xi_1, \xi_2) - G_-(\xi_2, \xi_1)$$

and  $\langle G_-^a, h_\gamma \otimes h_\delta \rangle = \int_{\Sigma_- \times \Sigma_-} \overline{G_-^a(\xi_1, \xi_2)} h_\gamma(\xi_1) h_\delta(\xi_2) d\xi_1 d\xi_2$ .

In a complete similar way, we also get

$$\begin{aligned} A_{+-} \Phi^{(\ell, m, n)} &= (-1)^m \sum_{\alpha=1}^m \sum_{\beta=1}^n (-1)^{\alpha+\beta} \langle G_{+-}, g_\alpha \otimes h_\beta \rangle \\ &\quad \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha}} b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n\} \\ k \neq \beta}} b_-^*(h_k) \Omega \end{aligned} \tag{14}$$

$$A_{-+}\Phi^{(\ell,m,n)} = (-1)^m \sum_{\alpha=1}^m \sum_{\beta=1}^n (-1)^{\alpha+\beta} \langle G_{-+}, h_\beta \otimes g_\alpha \rangle$$

$$\prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha}} b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n\} \\ k \neq \beta}} b_-^*(h_k) \Omega$$

where

$$\langle G_{+-}, g_\alpha \otimes h_\beta \rangle = \int_{\Sigma_+ \times \Sigma_-} \overline{G_{+-}(\xi_1, \xi_2)} g_\alpha(\xi_1) h_\beta(\xi_2) d\xi_1 d\xi_2$$

and

$$\langle G_{-+}, h_\beta \otimes g_\alpha \rangle = \int_{\Sigma_- \times \Sigma_+} \overline{G_{-+}(\xi_1, \xi_2)} h_\beta(\xi_1) g_\alpha(\xi_2) d\xi_1 d\xi_2.$$

Using (14) we get, as for (12)

$$A_{-+}\Phi^{(\ell,m,n)} = \sum_{\alpha=1}^m \sum_{\beta=1}^n (-1)^{\alpha+\beta} \sum_{1 \leq \gamma < \beta \leq n} (-1)^{\gamma+1} \langle G_{-+}^a, h_\gamma \otimes g_\alpha \otimes h_\beta \rangle$$

$$\prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha}} b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n\} \\ k \neq \beta, k \neq \gamma}} b_-^*(h_k) \Omega \quad (15)$$

and

$$A_{+-}\Phi^{(\ell,m,n)} = \sum_{\alpha=1}^m \sum_{\beta=1}^n (-1)^{\alpha+\beta} \sum_{1 \leq \gamma < \beta \leq n} (-1)^{\gamma+1} \langle G_{+-}^a, g_\gamma \otimes h_\alpha \otimes g_\beta \rangle$$

$$\prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \gamma, j \neq \beta}} b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n\} \\ k \neq \alpha}} b_-^*(h_k) \Omega$$

where

$$G_{\epsilon\epsilon'\epsilon}^a(\xi_1, \xi_2, \xi_3) = G_{\epsilon\epsilon'\epsilon}(\xi_1, \xi_2, \xi_3) - G_{\epsilon\epsilon'\epsilon}(\xi_3, \xi_2, \xi_1)$$

and

$$\langle G_{-+}^a, h_\gamma \otimes g_\alpha \otimes h_\beta \rangle = \int_{\Sigma_- \times \Sigma_+ \times \Sigma_-} \overline{G_{-+}^a(\xi_1, \xi_2, \xi_3)} h_\gamma(\xi_1) g_\alpha(\xi_2) h_\beta(\xi_3) d\xi_1 d\xi_2 d\xi_3$$

$$\langle G_{+-}^a, g_\gamma \otimes h_\alpha \otimes g_\beta \rangle = \int_{\Sigma_+ \times \Sigma_- \times \Sigma_+} \overline{G_{+-}^a(\xi_1, \xi_2, \xi_3)} g_\gamma(\xi_1) h_\alpha(\xi_2) g_\beta(\xi_3) d\xi_1 d\xi_2 d\xi_3$$

By (10) and (13), we finally obtain

$$\begin{aligned}
A_{++--}\Phi^{(\ell,m,n)} = & \sum_{1 \leq \alpha < \beta \leq m} (-1)^{\alpha+\beta} \sum_{1 \leq \gamma < \delta \leq n} (-1)^{\gamma+\delta} \langle G_{++--}^{aa}, g_\alpha \otimes g_\beta \otimes h_\gamma \otimes h_\delta \rangle \\
& \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha, j \neq \beta}} b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n\} \\ k \neq \gamma, k \neq \delta}} b_-^*(h_k) \Omega
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
A_{--++}\Phi^{(\ell,m,n)} = & \sum_{1 \leq \gamma < \delta \leq n} (-1)^{\gamma+\delta} \sum_{1 \leq \alpha < \beta \leq m} (-1)^{\alpha+\beta} \langle G_{--++}^{aa}, h_\gamma \otimes h_\delta \otimes g_\alpha \otimes g_\beta \rangle \\
& \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha, j \neq \beta}} b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n\} \\ k \neq \gamma, k \neq \delta}} b_-^*(h_k) \Omega
\end{aligned}$$

where

$$\begin{aligned}
G_{\epsilon\epsilon\epsilon'\epsilon'}^{aa}(\xi_1, \xi_2, \xi_3, \xi_4) = & G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4) - G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_1, \xi_2, \xi_4, \xi_3) \\
& - G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_2, \xi_1, \xi_3, \xi_4) + G_{\epsilon\epsilon\epsilon'\epsilon'}(\xi_2, \xi_1, \xi_4, \xi_3).
\end{aligned}$$

We have the following proposition

**Proposition 3.4.** *We have, for  $\epsilon \neq \epsilon'$*

- (i)  $\|A_\epsilon\| = \|A_\epsilon^*\| \leq \|G_\epsilon^a\|_{L^2(\Sigma_\epsilon \times \Sigma_\epsilon)}$
- (ii)  $\|A_{\epsilon\epsilon'}\| = \|A_{\epsilon\epsilon'}^*\| \leq \|G_{\epsilon\epsilon'}\|_{L^2(\Sigma_\epsilon \times \Sigma_{\epsilon'})}$
- (iii)  $\|A_{\epsilon\epsilon'\epsilon}\| = \|A_{\epsilon\epsilon'\epsilon}^*\| \leq \|G_{\epsilon\epsilon'\epsilon}^a\|_{L^2(\Sigma_\epsilon \times \Sigma_{\epsilon'} \times \Sigma_\epsilon)}$
- (iv)  $\|A_{\epsilon\epsilon\epsilon'\epsilon'}\| = \|A_{\epsilon\epsilon\epsilon'\epsilon'}^*\| \leq \|G_{\epsilon\epsilon\epsilon'\epsilon'}^{aa}\|_{L^2(\Sigma_\epsilon \times \Sigma_\epsilon \times \Sigma_{\epsilon'} \times \Sigma_{\epsilon'})}$

**Remark 3.5.** *When we consider electrons in the discrete spectrum, the estimates of Proposition 3.4 are still true. We just have to substitute  $\Sigma_d$  for  $\Sigma_+$ .*

*Proof.* As mentioned above, for the proof in the case of electrons, we only consider electrons in the continuous spectrum. The proof for electrons in the discrete spectrum is similar and simpler. For the proof of (i) we restrict ourselves to  $A_+$ . The case

of  $A_-$  is similar. Since  $\left( \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha, j \neq \beta}} b_+^*(g_j) \otimes \prod_{k=1}^n b_-^*(h_k) \Omega \right)_{1 \leq \alpha < \beta \leq m}$  is an orthonormal family, it follows from (10) that

$$\|A_+ \Phi^{(\ell,m,n)}\|^2 = \sum_{1 \leq \alpha < \beta \leq m} |\langle G_+, g_\alpha \otimes g_\beta \rangle|_{L^2(\Sigma_+ \times \Sigma_+)}^2 \leq \|G_+^a\|_{L^2(\Sigma_+ \times \Sigma_+)}^2 \|\Phi^{(\ell,m,n)}\|^2.$$

In order to prove (i), it is enough to show that

$$\|A_+ \Psi\|^2 \leq \|G_+^a\|^2 \|\Psi\|^2 \quad (17)$$

for every  $\Psi \in \mathfrak{F}_{0,D}$ .

The most significant finite linear combination of basis vectors is the following one

$$\Psi = \sum_{\mu=1}^N \sum_{\nu=1}^P \lambda_{\mu\nu} \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^{m-1} b_+^*(g_j) b_+^*(g_{m_\mu}) \otimes \prod_{k=1}^{n-1} b_-^*(h_k) b_-^*(h_{n_\nu}) \Omega \quad (18)$$

Here  $N$  and  $P$  are positive integers and we have  $m-1 < m_1 < m_2 < \dots < m_N$  and  $n-1 < n_1 < n_2 < \dots < n_P$ . From now on we restrict ourselves to finite linear combination of the form (18). The proof of inequality (17) is easier for any other finite linear combination of basis vectors. Note that  $\|\Psi\|^2 = \sum_{\mu=1}^N \sum_{\nu=1}^P |\lambda_{\mu\nu}|^2$ . By (10) we get

$$\begin{aligned} A_+ \Psi &= \sum_{\mu=1}^N \sum_{\nu=1}^P \sum_{1 \leq \alpha < \beta \leq m-1} (-1)^{\alpha+\beta} \lambda_{\mu\nu} \langle G_+^a, g_\alpha \otimes g_\beta \rangle \\ &\quad \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m-1\} \\ j \neq \alpha, j \neq \beta}} b_+^*(g_j) b_+^*(g_{m_\mu}) \otimes \prod_{k=1}^{n-1} b_-^*(h_k) b_-^*(h_{n_\nu}) \Omega \\ &\quad + \sum_{\nu=1}^P \sum_{1 \leq \alpha \leq m-1} (-1)^{\alpha+m} \left( \sum_{\mu=1}^N \lambda_{\mu\nu} \langle G_+^a, g_\alpha \otimes g_{m_\mu} \rangle \right) \\ &\quad \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m-1\} \\ j \neq \alpha}} b_+^*(g_j) \otimes \prod_{k=1}^{n-1} b_-^*(h_k) b_-^*(h_{n_\nu}) \Omega. \end{aligned} \quad (19)$$

The right hand side of (19) is a linear combination of vectors of an orthogonal family. Thus

$$\begin{aligned} \|A_+ \Psi\|^2 &= \sum_{1 \leq \alpha < \beta \leq m-1} \left( \sum_{\nu=1}^P \sum_{\mu=1}^N |\lambda_{\mu\nu}|^2 \right) |\langle G_+^a, g_\alpha \otimes g_\beta \rangle|^2 \\ &\quad + \sum_{1 \leq \alpha \leq m-1} \sum_{\nu=1}^P \left| \sum_{\mu=1}^N \lambda_{\mu\nu} \langle G_+^a, g_\alpha \otimes g_{m_\mu} \rangle \right|^2 \end{aligned}$$

By the Cauchy-Schwarz inequality we get

$$\begin{aligned} \|A_+ \Psi\|^2 &\leq (\sum_{\mu=1}^N \sum_{\nu=1}^P |\lambda_{\mu\nu}|^2) \left[ \sum_{1 \leq \alpha < \beta \leq m-1} |\langle G_+^a, g_\alpha \otimes g_\beta \rangle|^2 \right. \\ &\quad \left. + \sum_{1 \leq \alpha \leq m-1} \sum_{\mu=1}^N |\langle G_+^a, g_\alpha \otimes g_{m\mu} \rangle|^2 \right] \leq \|G_+^a\|^2 \|\Psi\|^2. \end{aligned}$$

This concludes the proof of (i). Since

$$\left( \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha}} b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n\} \\ k \neq \beta}} b_+^*(h_k) \Omega \right)_{\substack{1 \leq \alpha \leq m \\ 1 \leq \beta \leq n}}$$

is an orthonormal family we have

$$\|A_{+-} \Phi^{(\ell, m, n)}\|^2 = \sum_{\alpha=1}^m \sum_{\beta=1}^n |\langle G_{+-}, g_\alpha \otimes h_\beta \rangle|^2 \leq \|G_{+-}\|^2 \|\Phi^{(\ell, m, n)}\|^2$$

Let  $\Psi$  be of the form (18). By (14) we get

$$\begin{aligned}
A_{+-}\Psi &= (-1)^m \sum_{\mu=1}^N \sum_{\nu=1}^P \sum_{\alpha=1}^{m-1} \sum_{\beta=1}^{n-1} (-1)^{\alpha+\beta} \lambda_{\mu\nu} \langle G_{+-}, g_\alpha \otimes h_\beta \rangle \\
&\quad \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m-1\} \\ j \neq \alpha}} b_+^*(g_j) b_+^*(g_{m_\mu}) \otimes \prod_{\substack{k \in \{1, \dots, n-1\} \\ k \neq \beta}} b_-^*(h_k) b_-^*(h_{n_\nu}) \Omega \\
&\quad + (-1)^m \sum_{\mu=1}^N \sum_{\beta=1}^{n-1} (-1)^\beta \sum_{\nu=1}^P \lambda_{\mu\nu} \langle G_{+-}, g_{m_\mu} \otimes h_\beta \rangle \\
&\quad \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^{m-1} b_+^*(g_j) \otimes \prod_{\substack{k \in \{1, \dots, n-1\} \\ k \neq \beta}} b_-^*(h_k) b_-^*(h_{n_\nu}) \Omega \\
&\quad + (-1)^{m+n} \sum_{\nu=1}^P \sum_{\alpha=1}^{m-1} (-1)^\alpha \sum_{\mu=1}^N \lambda_{\mu\nu} \langle G_{+-}, g_\alpha \otimes h_{n_\nu} \rangle \\
&\quad \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{\substack{j \in \{1, \dots, m-1\} \\ j \neq \alpha}} b_+^*(g_j) b_+^*(g_{m_\mu}) \otimes \prod_{k=1}^{n-1} b_-^*(h_k) \Omega \\
&\quad + (-1)^n \left( \sum_{\mu=1}^N \sum_{\nu=1}^P \lambda_{\mu\nu} \langle G_{+-}, g_{m_\mu} \otimes h_{n_\nu} \rangle \right) \prod_{i=1}^{\ell} b_d^*(f_i) \otimes \prod_{j=1}^{m-1} b_+^*(g_j) \otimes \prod_{k=1}^{n-1} b_-^*(h_k) \Omega
\end{aligned} \tag{20}$$

The right hand side of (20) is a linear combination of vectors of an orthogonal family. Thus we obtain

$$\begin{aligned}
&\|A_{+-}\Psi\|^2 \\
&= \sum_{\mu=1}^N \sum_{\nu=1}^P \sum_{\alpha=1}^{m-1} \sum_{\beta=1}^{n-1} |\lambda_{\mu\nu}|^2 |\langle G_{+-}, g_\alpha \otimes h_\beta \rangle|^2 + \sum_{\beta=1}^{n-1} \sum_{\nu=1}^P \left| \sum_{\mu=1}^N \lambda_{\mu\nu} \langle G_{+-}, g_{m_\mu} \otimes h_\beta \rangle \right|^2 \\
&\quad + \sum_{\alpha=1}^{m-1} \sum_{\mu=1}^N \left| \sum_{\nu=1}^P \lambda_{\mu\nu} \langle G_{+-}, g_\alpha \otimes h_{n_\nu} \rangle \right|^2 + \left| \sum_{\mu=1}^N \sum_{\nu=1}^P \lambda_{\mu\nu} \langle G_{+-}, g_{m_\mu} \otimes h_{n_\nu} \rangle \right|^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality we get

$$\begin{aligned}
& \|A_{+-}\Psi\|^2 \\
& \leq \sum_{\mu=1}^N \sum_{\nu=1}^P |\lambda_{\mu\nu}|^2 \sum_{\alpha=1}^{m-1} \sum_{\beta=1}^{n-1} |\langle G_{+-}, g_\alpha \otimes h_\beta \rangle|^2 + \sum_{\beta=1}^{n-1} \sum_{\nu=1}^P \sum_{\mu=1}^N |\lambda_{\mu\nu}|^2 \sum_{\mu=1}^N |\langle G_{+-}, g_{m_\mu} \otimes h_\beta \rangle|^2 \\
& + \sum_{\alpha=1}^{m-1} \sum_{\mu=1}^N \sum_{\nu=1}^P |\lambda_{\mu\nu}|^2 \sum_{\nu=1}^P |\langle G_{+-}, g_\alpha \otimes h_{n_\nu} \rangle|^2 + \sum_{\mu=1}^N \sum_{\nu=1}^P |\lambda_{\mu\nu}|^2 \sum_{\mu=1}^N \sum_{\nu=1}^P |\langle G_{+-}, g_{m_\mu} \otimes h_{n_\nu} \rangle|^2 \\
& \leq \sum_{\mu=1}^N \sum_{\nu=1}^P |\lambda_{\mu\nu}|^2 \left[ \sum_{\alpha=1}^{m-1} \sum_{\beta=1}^{n-1} |\langle G_{+-}, g_\alpha \otimes h_\beta \rangle|^2 + \sum_{\beta=1}^{n-1} \sum_{\mu=1}^N |\langle G_{+-}, g_{m_\mu} \otimes h_\beta \rangle|^2 \right. \\
& \left. + \sum_{\alpha=1}^{m-1} \sum_{\nu=1}^P |\langle G_{+-}, g_\alpha \otimes h_{n_\nu} \rangle|^2 + \sum_{\mu=1}^N \sum_{\nu=1}^P |\langle G_{+-}, g_{m_\mu} \otimes h_{n_\nu} \rangle|^2 \right] \leq \|G_{+-}\|^2 \|\Psi\|^2.
\end{aligned}$$

The proof for  $A_{-+}$  is the same. This concludes the proof of (ii).

Using (15) and (16) and following the same method as for proving (i) and (ii), we prove (iii) and (iv). See [13] for details.

Proposition 3.4 is thus proved.  $\square$

**Remark 3.6.** *The following formal operators*

$$\begin{aligned}
& \int d\xi_1 d\xi_2 G_\epsilon(\xi_1, \xi_2) b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2), \\
& \int d\xi_1 d\xi_2 G_{\epsilon\epsilon'}(\xi_1, \xi_2) b_\epsilon^*(\xi_1) b_{\epsilon'}^*(\xi_2), \\
& \int d\xi_1 d\xi_2 d\xi_3 G_{\epsilon\epsilon'\epsilon}(\xi_1, \xi_2, \xi_3) b_\epsilon^*(\xi_1) b_{\epsilon'}^*(\xi_2) b_\epsilon^*(\xi_3),
\end{aligned}$$

and

$$\int d\xi_1 d\xi_2 d\xi_3 d\xi_4 G_{\epsilon\epsilon'\epsilon'}(\xi_1, \xi_2, \xi_3, \xi_4) b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_{\epsilon'}^*(\xi_3) b_{\epsilon'}^*(\xi_4),$$

are the ones associated respectively with  $A_\epsilon^*$ ,  $A_{\epsilon,\epsilon'}^*$ ,  $A_{\epsilon,\epsilon',\epsilon}^*$  and  $A_{\epsilon\epsilon'\epsilon'}^*$ , as quadratic forms on  $\mathfrak{F}_{0,D} \times \mathfrak{F}_{0,D}$ .

We now investigate operators in  $\mathfrak{F}_D$  built from a product of creation and annihilation operators. Let us introduce

$$\begin{aligned}
\mathfrak{F}_{D,fin} = \Big\{ \Psi = (\Psi^{(q,r,s)})_{q \geq 0, r \geq 0, s \geq 0}; & \text{ } \Psi^{(q,r,s)} \text{ is in the Schwartz space and} \\
& \Psi^{(q,r,s)} = 0 \text{ for all but finitely many } (q, r, s) \Big\}.
\end{aligned}$$

$\mathfrak{F}_{D,fin}$  is a core for  $N_D$ . For  $G_\epsilon \in L^2$ , by mimicking the proof of [35, Theorem X.44], one can show that there exists two operators, denoted by  $B_\epsilon$  ( $\epsilon \in \{+, -\}$ ), such that  $B_\epsilon$  are the unique operators in  $\mathfrak{F}_D$  such that  $\mathfrak{F}_{D,fin} \subset \mathfrak{D}(B_\epsilon)$  is a core for  $B_\epsilon$  and

$$B_\epsilon = \int_{\Sigma_\epsilon \times \Sigma_\epsilon} d\xi_1 d\xi_2 G_\epsilon(\xi_1, \xi_2) b_\epsilon^*(\xi_1) b_\epsilon(\xi_2)$$

and

$$B_\epsilon^* = - \int_{\Sigma_\epsilon \times \Sigma_\epsilon} d\xi_1 d\xi_2 \overline{G_\epsilon(\xi_1, \xi_2)} b_\epsilon^*(\xi_1) b_\epsilon(\xi_2) \quad (21)$$

as quadratic forms on  $\mathfrak{F}_{D,fin} \times \mathfrak{F}_{D,fin}$ . Similarly to (21), for  $G_{d,\epsilon} = (G_{\gamma,n;\epsilon})_{(\gamma,n) \in \mathcal{D}_d} \in L^2(\mathcal{D}_d \times \Sigma_\epsilon)$  and  $G_{d,d} = (G_{\gamma,n;\gamma',n'})_{(\gamma,n;\gamma',n') \in \mathcal{D}_d \times \mathcal{D}_d} \in L^2(\mathcal{D}_d \times \mathcal{D}_d)$ , we define the operators

$$B_{d,\epsilon} = \sum_{\gamma,n} \int_{\Sigma_\epsilon} d\xi G_{\gamma,n;\epsilon}(\xi) b_{\gamma,n}^* b_\epsilon(\xi)$$

and

$$B_{d,d} = \sum_{\gamma,n} \sum_{\gamma',n'} G_{\gamma,n;\gamma',n'} b_{\gamma,n}^* b_{\gamma',n'}$$

We then have the following proposition whose proof is borrowed from [20]

**Proposition 3.7.** *For  $G_\epsilon \in L^2(\Sigma_\epsilon \times \Sigma_\epsilon)$ ,  $G_{d,\epsilon} \in L^2(\mathcal{D}_d \times \Sigma_\epsilon)$  and  $G_{d,d} \in L^2(\mathcal{D}_d \times \mathcal{D}_d)$  we have*

$$\|B_\epsilon(N_D + 1)^{-1/2}\| \leq \|G_\epsilon\|_{L^2(\Sigma_\epsilon \times \Sigma_\epsilon)}, \quad (22)$$

$$\|(N_D + 1)^{-1/2} B_\epsilon\| \leq \|G_\epsilon\|_{L^2(\Sigma_\epsilon \times \Sigma_\epsilon)}, \quad (23)$$

$$\|B_{d,\epsilon}\| \leq \|G_{d,\epsilon}\|_{L^2(\mathcal{D}_d \times \Sigma_\epsilon)}, \quad (24)$$

and

$$\|B_{d,d}\| \leq \|G_{d,d}\|_{L^2(\mathcal{D}_d \times \mathcal{D}_d)}, \quad (25)$$

*Proof.* Since  $N_D$  is a self-adjoint operator, (23) follows from (21) and (22). We only investigate  $B_+$  since the proof for  $B_-$  is the same.

Let  $\Psi = (\Psi^{(q,r,s)})$  and  $\Phi = (\Phi^{(q',r',s')})$  be two vectors in  $\mathfrak{F}_{D,fin}$ . We have

$$\begin{aligned} & \langle \Phi^{(q',r',s')}, B_+ \Psi^{(q,r,s)} \rangle_{\mathfrak{F}_D} \\ &= \delta_{qq'} \delta_{rr'} \delta_{ss'} \int_{\Sigma_+ \times \Sigma_+} G_+(\xi_1, \xi_2) \langle b_+(\xi_1) \Phi^{(q,r,s)}, b_+(\xi_2) \Psi^{(q,r,s)} \rangle_{\mathfrak{F}_a^{(q,r-1,s)}} d\xi_1 d\xi_2 \end{aligned}$$

Thus  $B_+ \Psi^{(q,r,s)} \in \mathfrak{F}_a^{(q,r,s)}$  for every triple  $(q, r, s)$ . Therefore, we only need to estimate  $\langle \Phi^{(q,r,s)}, B_+ \Psi^{(q,r,s)} \rangle$  for every triple  $(q, r, s)$ . By the Fubini theorem we have

$$|\langle \Phi^{(q,r,s)}, B_+ \Psi^{(q,r,s)} \rangle|^2 = \left| \int_{\Sigma_+} \left\langle b_+(\xi_1) \Phi^{(q,r,s)}, \int_{\Sigma_+} G_+(\xi_1, \xi_2) b_+(\xi_2) \Psi^{(q,r,s)} d\xi_2 \right\rangle d\xi_1 \right|^2$$

By the Cauchy-Schwarz inequality and the fact that  $\|b_+(f)\| = \|f\|$ , we get

$$|\langle \Phi^{(q,r,s)}, B_+ \Psi^{(q,r,s)} \rangle|^2 \leq \left( \int_{\Sigma_+} \|b_+(\xi_1) \Phi^{(q,r,s)}\| \left( \int_{\Sigma_+} |G_+(\xi_1, \xi_2)|^2 d\xi_2 \right)^{\frac{1}{2}} d\xi_1 \right)^2 \|\Psi^{(q,r,s)}\|^2$$

Applying again Cauchy-Schwarz inequality and the definition of  $b_+(\xi)$ , we finally get

$$\begin{aligned} |\langle \Phi^{(q,r,s)}, B_+ \Psi^{(q,r,s)} \rangle|^2 &\leq r \|G_+\|^2 \|\Phi^{(q,r,s)}\|^2 \|\Psi^{(q,r,s)}\|^2 \\ &\leq \|G_+\|^2 \|\Phi^{(q,r,s)}\|^2 \|(N_D + 1)^{\frac{1}{2}} \Psi^{(q,r,s)}\|^2. \end{aligned} \quad (26)$$

Since  $B_+ \Psi^{(q,r,s)} \in \mathfrak{F}_a^{(q,r,s)}$ , we have

$$|\langle \Phi, B_+ \Psi^{(q,r,s)} \rangle|^2 \leq \|G_+\|^2 \|\Phi\|^2 \|(N_D + 1)^{\frac{1}{2}} \Psi^{(q,r,s)}\|^2, \quad (27)$$

for every  $\Phi \in \mathfrak{F}_{D,fin}$ . Now, since  $\mathfrak{F}_{D,fin}$  is dense in  $\mathfrak{F}_D$ , inequality (27) holds for all  $\Phi \in \mathfrak{F}_D$  and all triples  $(q, r, s)$ . Therefore, we have

$$\|B_+ \Psi^{(q,r,s)}\|^2 \leq \|G_+\|^2 \|(N_D + 1)^{\frac{1}{2}} \Psi^{(q,r,s)}\|^2$$

which yields

$$\|B_+ \Psi\| \leq \|G_+\| \|(N_D + 1)^{\frac{1}{2}} \Psi\| \quad (28)$$

for every  $\Psi \in \mathfrak{F}_{D,fin}$ . Since  $\mathfrak{F}_{D,fin}$  is a core for  $(N_D + 1)^{\frac{1}{2}}$ ,  $\mathfrak{D}(B_+) \supset \mathfrak{D}((N_D + 1)^{\frac{1}{2}})$  and (28) is still true for every  $\Psi \in \mathfrak{D}((N_D + 1)^{\frac{1}{2}})$ .

The proof of (24) and (25) is similar to the above one, if we use in addition that for all  $\gamma, n$ ,  $\|b_{\gamma,n}^*\| = 1$ . This concludes the proof of Proposition 3.7.  $\square$

**Remark 3.8.** Inequalities (22) and (23) are the best estimates that we can get. Indeed, set

$$\Phi^{(l,m,n)} = \prod_{i=1}^l b_d^*(f_i) \otimes \prod_{j=1}^m b_+^*(g_j) \otimes \prod_{k=1}^n b_-^*(h_k) \Omega$$

we have

$$\begin{aligned} B_+ \Phi^{(l,m,n)} &= \sum_{\alpha=1}^m \langle G_+, g_\alpha \otimes g_\alpha \rangle \Phi^{(l,m,n)} \\ &+ \sum_{\alpha=1}^m (-1)^{\alpha+1} \sum_{\beta=m+1}^{\infty} \langle G_+, g_\beta \otimes g_\alpha \rangle \prod_{i=1}^l b_d^*(f_i) \otimes b_+^*(g_\beta) \prod_{\substack{j \in \{1, \dots, m\} \\ j \neq \alpha}} b_+^*(g_j) \otimes \prod_{k=1}^n b_-^*(h_k) \Omega. \end{aligned} \quad (29)$$

Two different vectors in the right hand side of (29) are orthogonal. Therefore, we get

$$\|B_+ \Phi^{(\ell,m,n)}\|^2 = \left| \sum_{\alpha=1}^m \langle G_+, g_\alpha \otimes g_\alpha \rangle \right|^2 + \sum_{\alpha=1}^m \sum_{\beta=m+1}^{\infty} |\langle G_+, g_\beta \otimes g_\alpha \rangle|^2$$

from which we deduce

$$\|B_+ \Phi^{(\ell,m,n)}\|^2 \leq m \|G_+\|^2 \|\Phi^{(\ell,m,n)}\|^2 \leq \|G_+\|^2 \|N_D^{\frac{1}{2}} \Phi^{(\ell,m,n)}\|^2.$$

We now have the following two lemmata whose proof is easy.

**Lemma 3.9.** *Let  $A$  (resp.  $B$ ) be a positive self-adjoint operator in the Hilbert space  $\mathfrak{H}_1$  (resp.  $\mathfrak{H}_2$ ) with domain  $\mathfrak{D}(A)$  (resp.  $\mathfrak{D}(B)$ ).*

*Then the operator  $H_0 = A \otimes \mathbb{1} + \mathbb{1} \otimes B$  with domain  $\mathfrak{D}(A \otimes \mathbb{1}) \cap \mathfrak{D}(\mathbb{1} \otimes B)$  is self-adjoint in  $\mathfrak{H}_1 \otimes \mathfrak{H}_2$  and we have*

$$\begin{aligned} \|(A \otimes \mathbb{1}) \psi\| &\leq \|H_0 \psi\|, \\ \|(\mathbb{1} \otimes B) \psi\| &\leq \|H_0 \psi\|, \end{aligned}$$

for every  $\psi \in \mathfrak{D}(A \otimes \mathbb{1}) \cap \mathfrak{D}(\mathbb{1} \otimes B)$ .

**Lemma 3.10.** *Let  $E_0 = \inf_{(\gamma,n) \in \mathcal{D}_d} E_{\gamma,n} > 0$ . We have*

$$E_0 \|N_D \Psi\| \leq \|d\Gamma(H_D) \Psi\|$$

for any  $\Psi \in \mathfrak{D}(d\Gamma(H_D))$ .

Applying Lemma 3.9 to  $A = d\Gamma(H_D)$  and  $B = H_{ph}$  we get, from Lemma 3.10, the following result

**Proposition 3.11.** *For any  $\Psi \in \mathfrak{D}(d\Gamma(H_D) \otimes \mathbb{1}) \cap \mathfrak{D}(\mathbb{1} \otimes H_{ph})$ , we have*

$$\begin{aligned} E_0 \| (N_D \otimes \mathbb{1}) \Psi \| &\leq \|H_0 \Psi\| \\ \| (\mathbb{1} \otimes H_{ph}) \Psi \| &\leq \|H_0 \Psi\| \end{aligned}$$

where  $H_0 = d\Gamma(H_D) \otimes \mathbb{1} + \mathbb{1} \otimes H_{ph}$

### 3.1 Proof of Theorem 2.2

The proof of Theorem 2.2 is achieved by showing that  $H_I^{(1)}$  and  $H_I^{(2)} \otimes \mathbb{I}$  are relatively  $H_0$ -bounded.

We first treat  $H_I^{(1)}$ . The result follows from Lemma 3.12 and Corollary 3.13. Then we study all the quartic terms that appear in  $H_I^2$ , and the results are collected in Corollary 3.17. In both cases, details of proofs are given in the Appendix.

The interaction between electrons, positrons and transversal photons can be written in the following form (See Section 2.4)

$$H_I^{(1)} = \sum_{i=1}^6 \sum_{\mu=1,2} \int d^3k (v_i^\mu(k) \otimes a_\mu^*(k) + v_i^{\mu*}(k) \otimes a_\mu(k)) \quad (30)$$

where

$$\begin{aligned} v_1^\mu(k) &= \sum_{\gamma,\gamma',n,\ell} G_{d,\gamma,\gamma',n,\ell}^\mu(k) b_{\gamma,n}^* b_{\gamma',\ell}, \\ v_2^\mu(k) &= \sum_{\gamma,\gamma',n} \int dp G_{d,+,\gamma,\gamma',n}^\mu(p; k) (b_{\gamma,n}^* b_{\gamma',+}(p) + b_{\gamma,+}^*(p) b_{\gamma',n}), \\ v_3^\mu(k) &= \sum_{\gamma,\gamma',n} \int dp G_{d,-,\gamma,\gamma',n}^\mu(p; k) (b_{\gamma,n}^* b_{\gamma',-}^*(p) + b_{\gamma,-}(p) b_{\gamma',n}), \\ v_4^\mu(k) &= \sum_{\gamma,\gamma'} \int \int dp dp' G_{+,-,\gamma,\gamma'}^\mu(p, p'; k) (b_{\gamma,+}^*(p) b_{\gamma',-}^*(p') + b_{\gamma,-}(p) b_{\gamma',+}(p')), \\ v_5^\mu(k) &= \sum_{\gamma,\gamma'} \int \int dp dp' G_{+,-,\gamma,\gamma'}^\mu(p, p'; k) b_{\gamma,+}^*(p) b_{\gamma',+}(p'), \\ v_6^\mu(k) &= \sum_{\gamma,\gamma'} \int \int dp dp' G_{-,-,\gamma,\gamma'}^\mu(p, p'; k) b_{\gamma,-}^*(p) b_{\gamma',-}(p'). \end{aligned}$$

It follows from Proposition 3.4, Proposition 3.7 that  $v_i^\mu(k)$ ,  $i \neq 5, 6$ ,  $v_j^\mu(k)(N_D + 1)^{-1/2}$ ,  $(N_D + 1)^{-1/2}v_j^\mu(k)$ ,  $j = 5, 6$  are bounded from  $\mathfrak{F}_D$  into  $\mathfrak{F}_D$ .

For  $\beta = 0, 1$ ,  $\mu = 1, 2$  and  $i \in \{1, 2, \dots, 6\}$ , we set

$$\begin{aligned} a_{\beta,i}^\mu &= \left( \int_{\mathbb{R}^3} \omega(k)^{-\beta} \|v_i^\mu(k)\|^2 d^3k \right)^{1/2}, \quad i \neq 5, 6 \\ a_{\beta,i}^\mu &= \left( \int_{\mathbb{R}^3} \omega(k)^{-\beta} \|v_i^\mu(k)(N_D + 1)^{-1/2}\|^2 d^3k \right)^{1/2}, \quad i = 5, 6. \end{aligned} \quad (31)$$

$$\begin{aligned} b_i^\mu &= \left( \int \omega(k)^{-1} \|v_i^\mu(k)\|^2 d^3k \right)^{1/2}, \quad i \neq 5, 6 \\ b_i^\mu &= \left( \int \omega(k)^{-1} \|(N_D + 1)^{-1/2} v_i^\mu(k)\|^2 d^3k \right)^{1/2}, \quad i = 5, 6. \end{aligned} \tag{32}$$

**Lemma 3.12.** *For every  $\Psi \in \mathfrak{D}(H_0)$ , we have*

$$\left\| \int d^3k v_i^\mu(k)^* \otimes a_\mu(k) \Psi \right\| \leq \frac{b_i^\mu}{\sqrt{E_0}} \|(H_0 + m_0 c^2) \Psi\|, \tag{33}$$

$$\begin{aligned} \left\| \int d^3k v_i^\mu(k) \otimes a_\mu^*(k) \Psi \right\|^2 &\leq \left( \frac{(a_{1,i}^\mu)^2}{E_0} + \varepsilon \frac{(a_{0,i}^\mu)^2}{E_0^2} \right) \|(H_0 + m_0 c^2) \Psi\|^2 \\ &\quad + \frac{(a_{0,i}^\mu)^2}{4\varepsilon} \|\Psi\|^2 \end{aligned} \tag{34}$$

for every  $\varepsilon > 0$ .

*Proof.* In the appendix, we prove that, for every  $\Psi \in \mathfrak{D}(H_0)$  and every  $\varepsilon > 0$ , we have

$$\left\| \int d^3k v_i^\mu(k)^* \otimes a_\mu(k) \Psi \right\| \leq b_i^\mu \|(N_D + 1)^{1/2} \otimes H_{ph}^{1/2} \Psi\|, \tag{35}$$

$$\begin{aligned} \left\| \int d^3k v_i^\mu(k) \otimes a_\mu^*(k) \Psi \right\|^2 &\leq (a_{1,i}^\mu)^2 \|(N_D + 1)^{1/2} \otimes H_{ph}^{1/2} \Psi\|^2 \\ &\quad + (a_{0,i}^\mu)^2 \left[ \varepsilon \|(N_D + 1) \otimes \mathbb{1} \Psi\|^2 + \frac{1}{4\varepsilon} \|\Psi\|^2 \right], \end{aligned} \tag{36}$$

which together with Lemma 3.9 and Lemma 3.10 give (33) and (34) by noting that  $E_0 < m_0 c^2$ .  $\square$

Finally, from (30), (33) and (34), we get

**Corollary 3.13.** *The operator  $H_I^{(1)}$  is  $(H_0 + m_0 c^2)$ -bounded, with relative bound*

$$C'_1 = \sum_{\mu=1,2} \sum_{i=1}^6 \frac{a_{1,i}^\mu + b_i^\mu}{\sqrt{E_0}}.$$

We now treat the term  $H_I^{(2)}$ . Again in the case of electrons, we restrict ourselves to the electrons in the continuous spectrum. The case of electrons in the discrete spectrum will be simpler to deal with.

For  $F^{(1)} \in L^2$ , the following formal operator

$$\int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4) b_+^*(\xi_1) b_-^*(\xi_2) b_+(\xi_3) b_-(\xi_4)$$

is defined as a quadratic form on  $\mathfrak{F}_{D,fin} \times \mathfrak{F}_{D,fin}$  in the following way

$$\int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \langle b_-(\xi_2) b_+(\xi_1) \Psi, b_+(\xi_3) b_-(\xi_4) \Phi \rangle$$

As for the quadratic terms, one can show that there exists an operator, denoted by  $C^{(1)}$  such that  $C^{(1)}$  is the unique operator in  $\mathfrak{F}_D$  such that  $\mathfrak{F}_{D,fin} \subset \mathfrak{D}(C^{(1)})$  is a core for  $C^{(1)}$  and

$$C^{(1)} = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4) b_+^*(\xi_1) b_-^*(\xi_2) b_+(\xi_3) b_-(\xi_4)$$

and

$$(C^{(1)})^* = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \overline{F^{(1)}(\xi_3, \xi_4, \xi_1, \xi_2)} b_+^*(\xi_1) b_-^*(\xi_2) b_+(\xi_3) b_-(\xi_4)$$

as quadratic forms on  $\mathfrak{F}_{D,fin} \times \mathfrak{F}_{D,fin}$ . In the same way, for  $F_\epsilon^{(2)}$ ,  $F_{\epsilon,\epsilon'}^{(3)}$  and  $F^{(4)}$  in  $L^2$ , we define the following operators as quadratic forms on  $\mathfrak{F}_{D,fin} \times \mathfrak{F}_{D,fin}$

$$C_\epsilon^{(2)} = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F_\epsilon^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4) b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_\epsilon(\xi_3) b_\epsilon(\xi_4),$$

$$(C_\epsilon^{(2)})^* = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \overline{F_\epsilon^{(2)}(\xi_4, \xi_3, \xi_2, \xi_1)} b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_\epsilon(\xi_3) b_\epsilon(\xi_4),$$

$$C_{\epsilon,\epsilon'}^{(3)} = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F_{\epsilon,\epsilon'}^{(3)}(\xi_1, \xi_2, \xi_3, \xi_4) b_\epsilon^*(\xi_1) b_\epsilon(\xi_2) b_{\epsilon'}(\xi_3) b_\epsilon(\xi_4),$$

$$(C_{\epsilon,\epsilon'}^{(3)})^* = - \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \overline{F_{\epsilon,\epsilon'}^{(3)}(\xi_4, \xi_2, \xi_3, \xi_1)} b_\epsilon^*(\xi_1) b_\epsilon^*(\xi_2) b_{\epsilon'}^*(\xi_3) b_\epsilon(\xi_4),$$

$$C^{(4)} = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F^{(4)}(\xi_1, \xi_2, \xi_3, \xi_4) b_+(\xi_1) b_+(\xi_2) b_-(\xi_3) b_-(\xi_4),$$

$$(C^{(4)})^* = \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \overline{F^{(4)}(\xi_1, \xi_2, \xi_3, \xi_4)} b_+^*(\xi_1) b_+^*(\xi_2) b_-^*(\xi_3) b_-^*(\xi_4),$$

We now have the following proposition

**Proposition 3.14.**

(i) For  $F^{(1)} \in L^2$  we have

$$\|C^{(1)}(N_D + 1)^{-1}\| \leq \frac{1}{\sqrt{2}} \|F^{(1)}\|$$

and

$$\|(C^{(1)})^*(N_D + 1)^{-1}\| \leq \frac{1}{\sqrt{2}} \|F^{(1)}\|$$

(ii) For  $F_\epsilon^{(2)} \in L^2$  we have

$$\|C_\epsilon^{(2)}(N_D + 1)^{-1}\| \leq \|F_\epsilon^{(2),a}\|$$

and

$$\|(C_\epsilon^{(2)})^*(N_D + 1)^{-1}\| \leq \|F_\epsilon^{(2),a}\|$$

where  $F_\epsilon^{(2),a}(\xi_1, \xi_2, \xi_3, \xi_4) = F_\epsilon^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4) - F_\epsilon^{(2)}(\xi_1, \xi_2, \xi_4, \xi_3)$ .

(iii) For  $F_{\epsilon,\epsilon'}^{(3)} \in L^2$  we have

$$\|C_{\epsilon,\epsilon'}^{(3)}(N_D + 1)^{-\frac{1}{2}}\| \leq \|F_{\epsilon,\epsilon'}^{(3),a}\|$$

and

$$\|(C_{\epsilon,\epsilon'}^{(3)})^*(N_D + 1)^{-\frac{1}{2}}\| \leq \|F_{\epsilon,\epsilon'}^{(3),a}\|$$

where  $F_{\epsilon,\epsilon'}^{(3),a}(\xi_1, \xi_2, \xi_3, \xi_4) = F_{\epsilon,\epsilon'}^{(3)}(\xi_1, \xi_2, \xi_3, \xi_4) - F_{\epsilon,\epsilon'}^{(3)}(\xi_1, \xi_4, \xi_3, \xi_2)$ .

(iv) For  $F^{(4)} \in L^2$  we have

$$\|C^{(4)}\| \leq \|F^{(4),aa}\|$$

and

$$\|(C^{(4)})^*\| \leq \|F^{(4),aa}\|$$

where  $F^{(4),aa}(\xi_1, \xi_2, \xi_3, \xi_4) = F^{(4)}(\xi_1, \xi_2, \xi_3, \xi_4) - F^{(4)}(\xi_1, \xi_2, \xi_4, \xi_3) - F^{(4)}(\xi_2, \xi_1, \xi_3, \xi_4) + F^{(4)}(\xi_2, \xi_1, \xi_4, \xi_3)$ .

**Remark 3.15.** The estimates of Proposition 3.14 are satisfied both for electrons in the continuous spectrum and the discrete one with appropriate  $L^2$ -norms.

*Proof.* As mentioned above, for the proof in the case of electrons, we only consider electrons in the continuous spectrum. The proof for electrons in the discrete spectrum is similar and simpler. We mimick the proof of Proposition 3.7 by using Proposition 3.4. We first prove (i). Let  $\Psi = (\Psi^{(q,r,s)})$  and  $\Phi = (\Phi^{(q,r,s)})$  be two vectors of  $\mathfrak{F}_{D,fin}$ . We have

$$\begin{aligned} & \langle \Phi^{(q',r',s')}, C^{(1)} \Psi^{(q,r,s)} \rangle = \delta_{q,q'} \delta_{r,r'} \delta_{s,s'} \\ & \times \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4) \langle b_-(\xi_2) b_+(\xi_1) \Phi^{(q,r,s)}, b_+(\xi_3) b_-(\xi_4) \Psi^{(q,r,s)} \rangle \end{aligned}$$

Thus  $C^{(1)}\Psi^{(q,r,s)} \in \mathfrak{F}_a^{(q,r,s)}$  for every triple  $(q, r, s)$ . By the Fubini Theorem we get

$$|\langle \Phi^{(q',r',s')}, C^{(1)}\Psi^{(q,r,s)} \rangle| = \left| \int \left\langle b_-(\xi_2)b_+(\xi_1)\Psi^{(q,r,s)}, \int F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4)b_+(\xi_3)b_-(\xi_4)\Psi^{(q,r,s)} d\xi_3 d\xi_4 \right\rangle d\xi_1 d\xi_2 \right|$$

By the Cauchy-Schwarz inequality and Proposition 3.4(ii) we get

$$|\langle \Phi^{(q,r,s)}, C^{(1)}\Psi^{(q,r,s)} \rangle|^2 \leq \left( \int \|b_-(\xi_2)b_+(\xi_1)\Psi^{(q,r,s)}\| \left( \int |F^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4)|^2 d\xi_3 d\xi_4 \right)^{\frac{1}{2}} d\xi_1 d\xi_2 \right)^2 \|\Psi^{(q,r,s)}\|^2$$

Again by the Cauchy-Schwarz inequality and the definitions of  $b_+(\xi)$  and  $b_-(\xi)$  we finally get

$$\begin{aligned} |\langle \Phi^{(q,r,s)}, C^{(1)}\Psi^{(q,r,s)} \rangle|^2 &\leq rs \|F^{(1)}\|^2 \|\Phi^{(q,r,s)}\|^2 \|\Psi^{(q,r,s)}\|^2 \\ &\leq \frac{1}{2} \|F^{(1)}\|^2 \|\Phi^{(q,r,s)}\|^2 \|(N_D + 1)^2 \Psi^{(q,r,s)}\|^2, \end{aligned}$$

which yields

$$\|C^{(1)}(N_D + 1)^{-1}\| \leq \frac{1}{\sqrt{2}} \|F^{(1)}\|_2.$$

The estimate for  $(C^{(1)})^*$  follows directly from above and from

$$\langle (C^{(1)})^* \Phi^{(q,r,s)}, \Psi^{(q,r,s)} \rangle = \langle (\Phi^{(q,r,s)}, C^{(1)}\Psi^{(q,r,s)}) \rangle$$

Let us now prove (ii). We consider only the case of  $C_+^{(2)}$ , since the proof for  $C_-^{(2)}$  is the same. We have

$$\begin{aligned} \langle \phi^{(q',r',s')}, C_+^{(2)}\Psi^{(q,r,s)} \rangle &= \delta_{qq'}\delta_{rr'}\delta_{ss'} \\ &\times \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F_+^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4) \langle b_+(\xi_2)b_+(\xi_1)\Phi^{(q,r,s)}, b_+(\xi_3)b_+(\xi_4)\Psi^{(q,r,s)} \rangle \end{aligned}$$

Thus  $C_+^{(2)}\Psi^{(q,r,s)} \in \mathfrak{F}_a^{(q,r,s)}$  for every triple  $(q, r, s)$  and

$$\begin{aligned} |\langle \Phi^{(q,r,s)}, C_+^{(2)}\Psi^{(q,r,s)} \rangle|^2 &= \left| \int \left\langle b_+(\xi_2)b_+(\xi_1)\Phi^{(q,r,s)}, \int F_+^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4)b_+(\xi_3)b_+(\xi_4)\Psi^{(q,r,s)} \right\rangle d\xi_1 d\xi_2 \right|^2 \end{aligned}$$

By the Cauchy-Schwarz inequality and Proposition 3.4(i) we obtain

$$\begin{aligned} |\langle \Phi^{(q,r,s)}, C_+^{(2)}\Psi^{(q,r,s)} \rangle|^2 &\leq \left( \int \|b_+(\xi_2)b_+(\xi_1)\Phi^{(q,r,s)}\| \left( \int |F_+^{(2),a}(\xi_1, \xi_2, \xi_3, \xi_4)|^2 d\xi_3 d\xi_4 \right)^{\frac{1}{2}} d\xi_1 d\xi_2 \right)^2 \|\Psi^{(q,r,s)}\|^2 \end{aligned}$$

Applying again Cauchy-Schwarz inequality and the definition of  $b_+(\xi)$  we get

$$\begin{aligned} |\langle \Phi^{(q,r,s)}, C_+^{(2)} \Psi^{(q,r,s)} \rangle|^2 &\leq r(r-1) \|F_+^{(2),a}\|^2 \|\Phi^{(q,r,s)}\|^2 \|\Psi^{(q,r,s)}\|^2 \\ &\leq \|F_+^{(2),a}\|^2 \|\Phi^{(q,r,s)}\|^2 \|(N_D + 1) \Psi^{(q,r,s)}\|^2. \end{aligned}$$

We conclude this item as in the proof of Proposition 3.7. We thus obtain

$$\|C_+^{(2)}(N_D + 1)^{-1}\| \leq \|F_\epsilon^{(2),a}\|.$$

The proof for  $C_-^{(2)}$ ,  $(C_+^{(2)})^*$  and  $(C_-^{(2)})^*$  is similar.

We now prove (iii). We have

$$\begin{aligned} \langle \Phi^{(q',r',s')}, C_{+-}^{(3)} \Psi^{(q,r,s)} \rangle &= \delta_{q',q} \delta_{r',r-1} \delta_{s',s-1} \\ &\times \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 F_{+-}^{(3)}(\xi_1, \xi_2, \xi_3, \xi_4) \langle b_+(\xi_1) \Phi^{(q',r',s')}, b_+(\xi_2) b_-(\xi_3) b_+(\xi_4) \Psi^{(q,r,s)} \rangle \end{aligned}$$

Thus  $C_{+-}^{(3)} \Psi^{(q,r,s)} \in \mathfrak{F}_a^{q,r-1,s-1}$  for  $r > 1$  and  $s > 1$  and  $C_{+-}^{(3)} \Psi^{(q,r,s)} = 0$  for  $r \leq 1$  or  $s \leq 1$ . We then have

$$\begin{aligned} |\langle \Phi^{(q,r-1,s-1)}, C_{+-}^{(3)} \Psi^{(q,r,s)} \rangle|^2 &= \left| \int \left\langle b_+(\xi_1) \Phi^{(q,r-1,s-1)}, \int F_{+-}^{(3)}(\xi_1, \xi_2, \xi_3, \xi_4) b_+(\xi_2) b_-(\xi_3) b_+(\xi_4) \Psi^{(q,r,s)} \right\rangle \right|^2 \end{aligned}$$

Using Cauchy-Schwarz inequality and Proposition 3.4(iii) we obtain

$$\begin{aligned} |\langle \Phi^{(q,r-1,s-1)}, C_{+-}^{(3)} \Psi^{(q,r,s)} \rangle|^2 &\leq \left( \int \|b_+(\xi_1) \Phi^{(q,r-1,s-1)}\| \left( \int |F_{+-}^{(3),a}(\xi_1, \xi_2, \xi_3, \xi_4)|^2 d\xi_2 d\xi_3 d\xi_4 \right)^{\frac{1}{2}} d\xi_1 \right)^2 \|\Psi^{(q,r,s)}\|^2 \end{aligned}$$

Thus, using Cauchy-Schwarz inequality and the definition of  $b_+(\xi)$  we get

$$\begin{aligned} |\langle \Phi^{(q,r-1,s-1)}, C_{+-}^{(3)} \Psi^{(q,r,s)} \rangle|^2 &\leq (r-1) \|F_{+-}^{(3),a}\|^2 \|\Phi^{(q,r-1,s-1)}\|^2 \|\Psi^{(q,r,s)}\|^2 \\ &\leq \|F_{+-}^{(3),a}\|^2 \|\Phi^{(q,r-1,s-1)}\|^2 \|(N_D + 1)^{\frac{1}{2}} \Psi^{(q,r,s)}\|^2 \end{aligned}$$

We conclude as in the proof of Proposition 3.7 that

$$\|C_{+-}^{(3)}(N_D + 1)^{-\frac{1}{2}}\| \leq \|F_{+-}^{(3),a}\|.$$

The proofs for  $C_{\epsilon\epsilon'}^{(3)}$  and  $(C_{\epsilon\epsilon'}^{(3)})^*$  are very similar.

Finally, by Proposition 3.4(iv) we get immediately

$$\|C^{(4)}\| \leq \|F^{(4),aa}\|$$

and

$$\|(C^{(4)})^*\| \leq \|F^{(4),aa}\|$$

This concludes the proof of Proposition 3.14.  $\square$

**Remark 3.16.** One can show, as in Remark 3.8 that the estimates (i), (ii), (iii) and (iv) are the best one can get.

As a consequence of Lemmas 3.9, 3.10 and Propositions 3.4, 3.7 and 3.14 we obtain

**Corollary 3.17.** For any  $\Psi \in \mathfrak{D}(N_D)$  we have

$$\begin{aligned} \|H_I^{(2)}\Psi\| &\leq \left( \frac{1}{\sqrt{2}} \|F^{(1)}\| + \|F_+^{(2),a}\| + \|F_-^{(2),a}\| \right) \|(N_D + 1)\Psi\| \\ &+ \left[ 2 \left( \|F_{+-}^{(3),a}\| + \|F_{-+}^{(3),a}\| \right) + \|F_+^{(5)}\| + \|F_-^{(5)}\| \right] \|(N_D + 1)^{\frac{1}{2}}\Psi\| \\ &+ 2 \left[ \|F^{(4)}\| + \|F_{+-}^{(6)}\| + \|F_{-+}^{(6)}\| \right] \|\Psi\|. \end{aligned}$$

*Proof of Theorem 2.2.* Supposing that  $C_0 < \infty$ ,  $C_1 < \infty$  and choosing  $g_1$  and  $g_2$  such that  $\frac{|g_1|}{\sqrt{E_0}}C_1 + \frac{|g_2|}{E_0}(\frac{1}{\sqrt{2}}\|F^{(1)}\| + \|F_+^{(2),a}\| + \|F_-^{(2),a}\|) < 1$ , Theorem 2.2 follows from Proposition 3.7, Corollary 3.13, Corollary 3.17 and Theorem V.4.3 of [29].

## 4 Proof of Theorem 2.3

Throughout this section we assume that the assumptions of Theorem 2.3 are satisfied. Let us introduce an infrared cutoff in the first part of the interaction

Let  $H_{I,m}^{(1)}$  be the operator obtained from (30) by substituting

$$v_{i,m}^\mu(k) := 1_{\{k; \omega(k) \geq m\}} v_i^\mu(k), \quad m > 0,$$

for  $v_i^\mu(k)$ . We then define

$$H_m := H_0 + g_1 H_{I,m}^{(1)} + g_2 H_I^{(2)} \otimes \mathbb{1}.$$

Theorem 2.3 will be a simple consequence of the following result.

**Theorem 4.1.** There exists  $g_0 > 0$  such that for every  $(g_1, g_2)$  satisfying  $|g_1| + |g_2| \leq g_0$ , the following properties hold.

(i) For every  $\Psi \in \mathfrak{D}(H_0)$ ,  $H_m \Psi \xrightarrow[m \rightarrow 0]{} H\Psi$ .

(ii) For every  $m \in (0, 1)$ ,  $H_m$  has a normalized ground state  $\Phi_m$ .

(iii) Fix  $\lambda$  in  $(E_0, m_0 c^2)$ . For every  $m > 0$ , we have

$$\langle \Phi_m, P_{(-\infty, \lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}} \Phi_m \rangle \geq 1 - \delta_{g_1, g_2}, \quad (37)$$

where  $\delta_{g_1, g_2}$  tends to zero when  $|g_1| + |g_2|$  tends to zero and  $\delta_{g_1, g_2} < 1$  for  $|g_1| + |g_2| \leq g_0$ .

Theorem 2.3 is easily deduced from Theorem 4.1 as follows: Let  $H_m$  and  $\Phi_m$  be the same as in Theorem 4.1. Since  $\|\Phi_m\| = 1$ , there exists a subsequence  $(\Phi_{m_k})$  of  $(\Phi_m)$  such that  $w - \lim \Phi_{m_k} = \Phi$ .

On the other hand, since  $P_{(-\infty, \lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}}$  is finite rank for  $\lambda \in (E_0, m_0 c^2)$ , it follows from (37) that

$$\langle \Phi, P_{(-\infty, \lambda]}(d\Gamma(H_D)) \otimes P_{\Omega_{ph}} \Phi \rangle \geq 1 - \delta_{g_1, g_2},$$

which implies that  $\Phi \neq 0$ .

Now, in order to conclude the proof of Theorem 2.3, it suffices to apply the following well known result [4]

**Lemma 4.2.** *Let  $T_n$ ,  $n \geq 1$ , and  $T$  be self-adjoint operators on a Hilbert space  $\mathfrak{K}$  having a common core  $\mathfrak{D}$  such that, for all  $\Phi \in \mathfrak{D}$ ,  $T_n \Phi \rightarrow T \Phi$  as  $n \rightarrow \infty$ . We assume that, for every  $n \geq 1$ ,  $T_n$  has a normalized ground state  $\Phi_n$  with ground state energy  $E_n$  such that  $\lim_{n \rightarrow \infty} E_n = E$  and  $\omega - \lim_{n \rightarrow +\infty} \Phi_n = \Phi \neq 0$ . Then  $\Phi$  is a ground state of  $T$  with ground state energy  $E$ .*

Theorem 4.1 will be proved in several steps. We first prove (i). We then prove (iii). Finally, by the periodic approximation method, we will prove the existence of a ground state for  $H_m$ ,  $m > 0$ .

## 4.1 Proof of (i) of Theorem 4.1

*Proof.* For  $m > 0$ , set

$$\tilde{v}_i^{\mu, m}(k) = 1_{\{k; \omega(k) < m\}}(k) v_i^\mu(k).$$

We denote by  $\tilde{a}_{\beta, j}^{\mu, m}$  (resp.  $\tilde{b}_j^{\mu, m}$ ) the expression (31) and (32) that we obtain by substituting  $\tilde{v}_j^{\mu, m}(k)$  for  $v_j^\mu(k)$ .

Since  $H - H_m = g_1(H_I^{(1)} - H_{I, m}^{(1)})$ , it follows from Lemma 3.12 that

$$\|(H - H_m)\Phi\| \leq |g_1| \sum_{\mu=1,2} \sum_{j=0}^6 \left( \left( \frac{\tilde{a}_{1,j}^{\mu, m} + \tilde{b}_j^{\mu, m}}{\sqrt{E_0}} + \frac{\tilde{a}_{0,j}^{\mu, m}}{E_0} \right) \|(H_0 + m_0 c^2)\Phi\| + \frac{\tilde{a}_{0,j}^{\mu, m}}{4} \|\Phi\| \right)$$

for every  $\Phi \in \mathfrak{D}(H_0)$ .

By Lebesgue's Theorem,  $(\tilde{a}_{1,j}^{\mu, m} + \tilde{a}_{0,j}^{\mu, m} + \tilde{b}_j^{\mu, m})$  tends to zero when  $m$  tends to zero. We then get, for every  $\Phi \in \mathfrak{D}(H_0)$ ,

$$\lim_{m \rightarrow 0} \|(H - H_m)\Phi\| = 0. \quad (38)$$

Thus the first statement of Theorem 4.1 is proved.  $\square$

## 4.2 Proof of (iii) of Theorem 4.1

The proof of (37) will be a consequence of the following lemmas.

**Lemma 4.3.** *For every  $m \in (0, 1)$ , we have*

$$E_m := \inf_{\substack{\Phi \in \mathfrak{D}(H_0) \\ \|\Phi\|=1}} (H_m \Phi, \Phi) \leq 0$$

and

$$\sup_{m \in (0, 1)} |E_m| < \infty. \quad (39)$$

*Proof.* Since  $\langle a_\mu(k) \Omega_{ph}, \Omega_{ph} \rangle = \langle \Omega_{ph}, a_\mu^*(k) \Omega_{ph} \rangle = 0$ , we have

$$\langle H_{I,m}^{(1)} \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph} \rangle = 0,$$

and

$$\langle (\mathbb{1} \otimes H_{ph}) \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph} \rangle = 0.$$

Furthermore we also obviously have

$$\langle H_I^{(2)} \Omega_D, \Omega_D \rangle = 0.$$

Hence we get

$$\langle H_m \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph} \rangle = \langle (d\Gamma(H_D) \otimes \mathbb{1}) \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph} \rangle = 0$$

and

$$E_m = \inf_{\substack{\Phi \in \mathfrak{D}(H_0) \\ \|\Phi\|=1}} \langle H_m \Phi, \Phi \rangle \leq \langle H_m \Omega_D \otimes \Omega_{ph}, \Omega_D \otimes \Omega_{ph} \rangle = 0$$

By Lemma 3.12, which also holds when  $v_i^\mu(k)$  is replaced by  $v_{i,m}^\mu(k)$ , we get for some finite constant  $C$  which does not depend on  $m \in (0, 1)$ :

$$\|H_{I,m}^{(1)} \Phi\| \leq C (\|H_0 + m_0 c^2\| \Phi + \|\Phi\|) \quad (40)$$

for every  $\Phi \in \mathfrak{D}(H_0)$ . From Corollary 3.17 we have, for some finite constant  $\tilde{C}$ :

$$\|(H_I^{(2)} \otimes \mathbb{1}) \Phi\| \leq \tilde{C} (\|H_0 + m_0 c^2\| \Phi + \|\Phi\|) \quad (41)$$

for every  $\Phi \in \mathfrak{D}(H_0)$ . Thus (40), (41) and the Kato-Rellich Theorem yield (39).  $\square$

In Section 4.3 below, we will show that, for  $|g_1| + |g_2|$  sufficiently small,  $H_m$  has a ground state  $\Phi_m(g_1, g_2)$ , i.e. there exists, for every  $m \in (0, 1)$ , a normalized solution to  $H_m \Phi_m(g_1, g_2) = E_m \Phi_m(g_1, g_2)$ .

We have

**Lemma 4.4.** *There exist  $\tilde{g}_0 > 0$  and  $\nu(\tilde{g}_0) > 0$ , independent of  $m \in (0, 1)$ , such that*

$$\|(H_0 + m_0 c^2)\Phi_m(g_1, g_2)\| \leq \nu(\tilde{g}_0),$$

for every  $m \in (0, 1)$  and for every  $(g_1, g_2)$  such that  $|g_1| + |g_2| \leq \tilde{g}_0$ .

*Proof.* For simplicity, we will drop the dependence on  $(g_1, g_2)$  in  $\Phi_m(g_1, g_2)$  by denoting  $\Phi_m := \Phi_m(g_1, g_2)$ . We have

$$\begin{aligned} H_0\Phi_m &= H_m\Phi_m - g_1 H_{I,m}^{(1)}\Phi_m - g_2(H_I^{(2)} \otimes \mathbb{1})\Phi_m \\ &= E_m\Phi_m - g_1 H_{I,m}^{(1)}\Phi_m - g_2(H_I^{(2)} \otimes \mathbb{1})\Phi_m. \end{aligned} \tag{42}$$

From (40), (41) and (42) we obtain

$$\|H_0\Phi_m\| \leq |E_m| + (|g_1|C + |g_2|\tilde{C}) [\|(H_0 + m_0 c^2)\Phi_m\| + 1],$$

which implies

$$\|(H_0 + m_0 c^2)\Phi_m\| \leq (m_0 c^2 + |E_m| + |g_1|C + |g_2|\tilde{C}) (1 - |g_1|C - |g_2|\tilde{C})^{-1}$$

To conclude, we apply Lemma 4.3, and we choose  $\tilde{g}_0 > 0$  such that  $|g_1|C + |g_2|\tilde{C} \leq 1/2$  for every  $(g_1, g_2)$  such that  $|g_1| + |g_2| \leq \tilde{g}_0$ .  $\square$

**Lemma 4.5.** *There exists  $C > 0$  such that*

$$\sum_{\mu=1,2} \| [1 \otimes a_\mu(k), H_{I,m}^{(1)}] \Phi \|^2 \leq C \frac{G(k)^2}{E_0^2} \|(H_0 + m_0 c^2)\Phi\|^2$$

for every  $\Phi \in \mathfrak{D}(H_0)$  and for a.e.  $k \in \mathbb{R}^3$ . Here

$$\begin{aligned} G(k)^2 &= \sum_{\mu=1,2} \sum_{\gamma, \gamma', n, \ell} |G_{d,\gamma,\gamma',n,l}^\mu(k)|^2 + 4 \sum_{\mu=1,2} \sum_{r=+,-} \sum_{\gamma, \gamma', n} \int |G_{d,r,\gamma,\gamma',n}^\mu(p; k)|^2 dp \\ &\quad + 4 \sum_{\mu=1,2} \sum_{\gamma, \gamma'} \int \int |G_{+, -, \gamma, \gamma'}(p, p'; k)|^2 dp dp' \\ &\quad + \sum_{\mu=1,2} \sum_{r=+,-} \sum_{\gamma, \gamma'} \int \int |G_{r,r,\gamma,\gamma'}(p, p'; k)|^2 dp dp' \end{aligned}$$

*Proof.* By using the commutation relations, a simple computation shows that

$$[\mathbb{1} \otimes a_\mu(k), H_{I,m}^{(1)}] = \sum_{i=1}^6 v_{i,m}^\mu(k)$$

The lemma now follows from Proposition 3.4 and Proposition 3.7.  $\square$

**Lemma 4.6.** *There exists  $C > 0$ , independent of  $g_1$  and  $g_2$  for  $|g_1| + |g_2| \leq \tilde{g}_0$ , such that*

$$(\mathbb{1} \otimes N_{ph}\Phi_m, \Phi_m) \leq C g_1^2 \left( \int_{\mathbb{R}^3} \frac{G(k)^2}{\omega(k)^2} d^3k \right),$$

for every  $m \in (0, 1)$ . Here

$$N_{ph} := \sum_{\mu=1,2} \int_{\mathbb{R}^3} d^3k a_\mu^*(k) a_\mu(k)$$

is the operator number of photons.

*Proof.* One easily checks that we have

$$[\mathbb{1} \otimes a_\mu(k), d\Gamma(H_D) \otimes \mathbb{1}] \Psi = 0,$$

$$[\mathbb{1} \otimes a_\mu(k), H_I^{(2)} \otimes \mathbb{1}] \Psi = 0,$$

and

$$[\mathbb{1} \otimes a_\mu(k), \mathbb{1} \otimes H_{ph}] \Psi = \omega(k) (\mathbb{1} \otimes a_\mu(k)) \Psi,$$

for every  $\Psi \in \mathfrak{D}(H_0)$  and for a.e.  $k \in \mathbb{R}^3$ .

We then obtain :

$$(H_m + \omega(k)) \mathbb{1} \otimes a_\mu(k) - \mathbb{1} \otimes a_\mu(k) H_m = g_1 [H_{I,m}^{(1)}, \mathbb{1} \otimes a_\mu(k)],$$

for a.e.  $k \in \mathbb{R}^3$ .

Applying this equality to  $\Phi_m$  and taking the scalar product with  $\mathbb{1} \otimes a_\mu(k) \Phi_m$ , we get

$$\|\mathbb{1} \otimes a_\mu(k) \Phi_m\| \leq \frac{|g_1|}{\omega(k)} \|[\mathbb{1} \otimes a_\mu(k), H_{I,m}] \Phi_m\| \quad (43)$$

for a.e.  $k \in \mathbb{R}^3$ .

Since

$$\langle \mathbb{1} \otimes N_{ph} \Phi_m, \Phi_m \rangle = \sum_{\mu=1,2} \int \|\mathbb{1} \otimes a_\mu(k) \Phi_m\|^2 d^3k$$

Lemma 4.6 follows from (43), Lemma 4.4 and Lemma 4.5.  $\square$

**Lemma 4.7.** *Fix  $\lambda$  in  $(0, m_0 c_0^2)$ . There exists  $\delta_{g_1, g_2}(\lambda) > 0$  such that  $\delta_{g_1, g_2}(\lambda)$  tends to zero when  $|g_1| + |g_2|$  tends to zero and*

$$\langle P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} \Phi_m, \Phi_m \rangle \leq \delta_{g_1, g_2}(\lambda)$$

for every  $m \in (0, 1)$ . Here  $P_D^\perp(\lambda) = \mathbb{1} - P_D(\lambda) = \mathbb{1}_{[\lambda, \infty)}(d\Gamma(H_D))$ .

*Proof.* Using  $P_{\Omega_{ph}} H_{ph} = 0$ , we get

$$\begin{aligned} & (P_D(\lambda)^\perp \otimes P_{\Omega_{ph}})(H_m - E_m) \\ &= P_D(\lambda)^\perp(d\Gamma(H_D) - E_m) \otimes P_{\Omega_{ph}} + g_1(P_D(\lambda)^\perp \otimes P_{\Omega_{ph}}) H_{I,m}^{(1)} \\ &\quad + g_2(P_D(\lambda)^\perp \otimes P_{\Omega_{ph}}) H_I^{(2)} \otimes \mathbb{1}. \end{aligned} \quad (44)$$

From (44) we obtain

$$\begin{aligned} 0 &= (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}})(H_m - E_m) \Phi_m \\ &= P_D^\perp(\lambda)(d\Gamma(H_D) - E_m) \otimes P_{\Omega_{ph}} \Phi_m + g_1 P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} H_{I,m}^{(1)} \Phi_m \\ &\quad + g_2(P_D(\lambda)^\perp \otimes P_{\Omega_{ph}}) H_I^{(2)} \otimes \mathbb{1}. \end{aligned} \quad (45)$$

Since

$$d\Gamma(H_D) P_D(\lambda)^\perp \geq \lambda P_D^\perp(\lambda),$$

by (45) and Lemma 4.3, we obtain

$$\begin{aligned} \langle P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} \Phi_m, \Phi_m \rangle &\leq -\lambda^{-1} \left[ g_1 \left\langle (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}}) H_{I,m}^{(1)} \Phi_m, \Phi_m \right\rangle \right. \\ &\quad \left. + g_2 \left\langle (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}})(H_I^{(2)} \otimes \mathbb{1}) \Phi_m, \Phi_m \right\rangle \right], \end{aligned}$$

for every  $m \in (0, 1)$ .

Lemma 4.7 then follows by remarking that Corollary 3.13 also holds for  $H_{I,m}^{(1)}$  and by using Lemma 3.17 and Lemma 4.4.  $\square$

We are now able to prove the inequality (37), i.e., point (iii) of Theorem 4.1.

*Proof.* The equation (37) is equivalent to

$$\langle (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} + \mathbb{1} \otimes P_{\Omega_{ph}}^\perp) \Phi_m, \Phi_m \rangle \leq \delta_{g_1, g_2},$$

Since  $P_{\Omega_{ph}}^\perp \leq N_{ph}$ , it suffices to show that

$$\langle (P_D^\perp(\lambda) \otimes P_{\Omega_{ph}} + \mathbb{1} \otimes N_{ph}) \Phi_m, \Phi_m \rangle \leq \delta_{g_1, g_2},$$

which follows from Lemma 4.6 and Lemma 4.7.

From the fact that  $\delta_{g_1, g_2}$  tends to zero when  $|g_1| + |g_2|$  tends to zero, we deduce the existence of  $\widetilde{g}_{00}$  such that  $\delta_{g_1, g_2} < 1$  for  $|g_1| + |g_2| \leq \widetilde{g}_{00}$ .  $\square$

### 4.3 Proof of (ii) of Theorem 4.1: Existence of a ground state for $H_m$

From now on we mimic the proof in [2] and [7] (See also [24], [25], [19] and [34]) and we use the same notations as in [9].

For  $m > 0$ , we set

$$\mathcal{A}_\ell = \{k \in \mathbb{R}^3 ; \omega(k) \geq m\}, \quad \mathcal{A}_s = \mathbb{R}^3 \setminus \mathcal{A}_\ell,$$

and

$$h_{s/\ell} = L^2(\mathcal{A}_{s/\ell}).$$

Let  $\mathfrak{F}(h_{\ell/s})$  be the bosonic Fock spaces associated with  $h_{\ell/s}$ , with associated vacua  $\Omega_{\ell/s}$ .

Using the following lemma (See [7] and [16]),

**Lemma 4.8.** *Let  $h_i$ ,  $i = 1, 2$  be two Hilbert spaces. There exists a unitary operator from the photonic Fock space over  $h_1 \oplus h_2$  to*

$$\mathfrak{F}_{ph}(h_1) \otimes \mathfrak{F}_{ph}(h_2).$$

We may identify  $H_{ph}$  (resp.  $g_1 H_I^{(1)}$ ) with

$$H_{ph,\ell} \otimes \mathbb{1}_s + \mathbb{1}_\ell \otimes H_{ph,s}, \quad (\text{resp. } g_1 H_{I,m}^{(1)} \otimes \mathbb{1}_s).$$

Here

$$H_{ph,\ell/s} = \sum_{\mu=1,2} \int_{\mathcal{A}_{\ell/s}} d^3 k \omega(k) a_\mu^*(k) a_\mu(k).$$

We set

$$H_{m,\ell} = d\Gamma(H_D) \otimes \mathbb{1}_s + \mathbb{1}_\ell \otimes H_{ph,\ell} + g_1 H_{I,m}^{(1)} + g_2 H_I^{(2)} \otimes \mathbb{1}_\ell.$$

In this representation,  $H_m$  appears as

$$H_m = H_{m,\ell} \otimes \mathbb{1}_s + \mathbb{1}_D \otimes \mathbb{1}_\ell \otimes H_{ph,s}. \quad (46)$$

We then have

**Lemma 4.9.**  *$H_{m,\ell}$  has a ground state  $\Phi_{m,\ell}$  if and only if  $H_m$  has a ground state  $\Phi_m = \Phi_{m,\ell} \otimes \Omega_s$ .*

*Proof.* In view of (46), one has

$$H_m(\Phi_{m,\ell} \otimes \Omega_s) = H_{m,\ell} \Phi_{m,\ell} \otimes \Omega_s \quad (47)$$

Furthermore, it results from (46) that

$$\sigma(H_m) = \overline{\sigma(H_{m,\ell}) + \sigma(H_{ph,s})} = [\inf \sigma(H_{m,\ell}), +\infty)$$

which, together with (47), gives the lemma.  $\square$

By Lemma 4.9, Theorem 4.1 then follows from

**Theorem 4.10.** *There exists  $\widetilde{g}_{000}$  such that, for every  $(g_1, g_2)$  such that  $|g_1| + |g_2| \leq \widetilde{g}_{000}$  and  $m \in (0, 1)$ ,  $H_{m,\ell}$  has a ground state  $\Phi_{m,\ell}$  at  $E_m = \inf \sigma(H_m) = \inf \sigma(H_{m,\ell})$ .*

*Proof.* Let  $\varepsilon > 0$  be a parameter. We decompose  $\mathbb{R}^3$  into a disjoint union of cubes of side length  $\varepsilon$ ,  $\mathbb{R}^3 = \bigcup_{n \in (\varepsilon \mathbb{Z}^3)} n + Q_\varepsilon$ , where  $Q_\varepsilon = [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^3$ . As in [9] and [34], for  $F \in L^1_{loc}(\mathbb{R}^3)$ , we define its  $\varepsilon$ -average by

$$\langle F \rangle_\varepsilon(k) = \varepsilon^{-3} \int_{n(k)+Q_\varepsilon} F(k') d^3 k',$$

where  $n(k) \in (\varepsilon \mathbb{Z})^3$  is such that  $k - n(k) \in Q_\varepsilon$ .

More generally, when  $G \in L^2(\Sigma \times \mathbb{R}_k^3)$ , we define its  $\varepsilon$ -average with respect to  $k$  by

$$\langle G \rangle_\varepsilon(k) := G_\varepsilon(\cdot, k) = \varepsilon^{-3} \int_{n(k)+Q_\varepsilon} G(\cdot, k') d^3 k'.$$

Let  $H_{I,m}^\varepsilon$  be the operator obtained from  $H_{I,m}^{(1)}$  by substituting  $\langle v_{i,m}^\mu \rangle_\varepsilon(k)$  for  $v_{i,m}^\mu(k)$ .

Similarly, we define  $H_{ph,\ell/s}^\varepsilon$  by substituting  $\langle \omega \rangle_\varepsilon(k)$  for  $\omega(k)$  in  $H_{ph,\ell/s}$ .

A simple calculation shows that

$$|\omega(k) - \langle \omega \rangle_\varepsilon(k)| \leq \tilde{C} \varepsilon \omega(k),$$

It implies

$$\pm(H_{ph,\ell} - H_{ph,\ell}^\varepsilon) \leq \tilde{C} \varepsilon H_{ph,\ell}. \quad (48)$$

Set  $H_{m,\ell}^\varepsilon = d\Gamma(H_D) \otimes \mathbb{1}_\ell + \mathbb{1}_D \otimes H_{ph,\ell}^\varepsilon + g_1 H_{I,m}^\varepsilon + g_2 H_I^{(2)} \otimes \mathbb{1}_l$ . Combining (48) with Corollary 3.13 and using the fact that

$$H_{m,\ell}^\varepsilon - H_{m,\ell} = \mathbb{1}_D \otimes H_{ph,\ell}^\varepsilon - \mathbb{1}_D \otimes H_{ph,\ell} + g_1 (H_{I,m}^\varepsilon - H_{I,m}^{(1)}),$$

we obtain

$$\|(H_{m,\ell}^\varepsilon - H_{m,\ell})(H_{m,\ell} - E_m + 1)^{-1}\| \leq \text{const.} \left( \varepsilon + \sum_{\beta=0,1} \sum_{j=1}^6 (a_{\beta,j}^{\mu,m,\varepsilon} + b_j^{\mu,m,\varepsilon}) \right), \quad (49)$$

where  $a_{\beta,j}^{\mu,m,\varepsilon}$  (resp.  $b_j^{\mu,m,\varepsilon}$ ) denotes the expression (31) (resp. (32)) that we obtain when we substitute  $v_{j,m}^\mu(k) - \langle v_{j,m}^\mu \rangle_\varepsilon(k)$  for  $v_j^\mu(k)$ .

Now, the right hand side of (49) converges to zero as  $\varepsilon \mapsto 0$ .

On the other hand, by mimicking [9] and [7] one shows that there exists  $\widetilde{g}_{000}$  such that the finite volume approximation  $H_{m,\ell}^\varepsilon$  has discrete spectrum in  $(-\infty, E_m + m_2)$ , for  $m_2 < m$  and for  $(g_1, g_2)$  such that  $|g_1| + |g_2| \leq \widetilde{g}_{000}$ .

Theorem 4.10 then follows from (49) and from the following lemma (see [26]).  $\square$

**Lemma 4.11.** *Let  $(T_n)_{n \geq 1}$  and  $T$  be bounded below self-adjoint operators on a Hilbert space. Suppose that*

i)  $T_n \rightarrow T$  as  $n \mapsto +\infty$  in the norm resolvent sense.

ii)  $T_n$  has purely discrete spectrum in  $[\inf \sigma(T_n), \inf \sigma(T_n) + C]$ , where  $C$  is a constant independent of  $n$ .

Then  $T$  has purely discrete spectrum in  $[\inf \sigma(T), \inf \sigma(T) + C]$ .

*Proofs of Theorem 2.3 and Theorem 4.1.* Choosing  $g_0 = \min\{\widetilde{g}_0, \widetilde{g}_{00}, \widetilde{g}_{000}\}$ , Theorem 4.1 follows from (38), Lemma 4.7 and Theorem 4.10. The proof of Theorem 2.3 is easily deduced from Theorem 4.1 as it is explained at the beginning of the Section 4.

## 5 Appendix

### 5.1 Proof of (35) and (36)

We keep the same notations as in Section 3.

**Lemma 5.1.** *Under the assumptions of Theorem 2.2, we have*

$$\left\| \int d^3k v_i^\mu(k)^* \otimes a_\mu(k) \Psi \right\| \leq b_i^\mu \|(N_D + 1)^{1/2} \otimes H_{ph}^{1/2} \Psi\|, \quad (50)$$

and

$$\begin{aligned} \left\| \int d^3k v_i^\mu(k) \otimes a_\mu^*(k) \Psi \right\|^2 &\leq (a_{1,i}^\mu)^2 \|(N_D + 1)^{1/2} \otimes H_{ph}^{1/2} \Psi\|^2 \\ &\quad + (a_{0,i}^\mu)^2 \left[ \varepsilon \|(N_D + 1) \otimes \mathbb{1} \Psi\|^2 + \frac{1}{4\varepsilon} \|\Psi\|^2 \right], \end{aligned} \quad (51)$$

for every  $\Psi \in \mathfrak{D}(H_0)$  and every  $\varepsilon > 0$ .

*Proof.* We only give the proof of (51) for  $i = 5$ . The case of  $i = 6$  is quite similar.

The other cases, i.e.  $i \neq 5, 6$  are simpler since  $v_i^\mu(k)$  is bounded on  $\mathfrak{F}_D$  for  $i \neq 5, 6$ . In the following we omit the indexes  $\mu$  and 5.

Set

$$a(v) = \int d^3k v^*(k) \otimes a(k)$$

$$a(v)^* = \int d^3k v(k) \otimes a^*(k)$$

Let  $u \in \mathfrak{D}(H_0)$ . For  $k \in \mathbb{R}^3$ , set

$$\Phi(k) = \omega(k)^{1/2}((N_D + 1)^{1/2} \otimes a(k)) u.$$

One has

$$\begin{aligned} \int \|\Phi(k)\|^2 d^3k &= \int \omega(k) \langle (N_D + 1)^{1/2} \otimes a(k)u, (N_D + 1)^{1/2} \otimes a(k)u \rangle d^3k \\ &= \left\langle (N_D + 1)^{1/2} \otimes \int \omega(k) a^*(k) a(k) d^3k u, (N_D + 1)^{1/2} \otimes \mathbb{1} u \right\rangle \\ &= \langle ((N_D + 1)^{1/2} \otimes H_{ph}) u, ((N_D + 1)^{1/2} \otimes \mathbb{1}) u \rangle \\ &= \|((N_D + 1)^{1/2} \otimes H_{ph}^{1/2}) u\|^2 \end{aligned} \quad (52)$$

Using (52), we get

$$\begin{aligned} \|a(v)u\|^2 &= \int \langle v^*(k) \otimes a(k)u, v^*(k') \otimes a(k')u \rangle d^3k d^3k' \\ &= \int \omega(k)^{-\frac{1}{2}} \omega(k')^{-\frac{1}{2}} \\ &\quad \times \left\langle v^*(k)((N_D + 1)^{-\frac{1}{2}} \otimes \mathbb{1}) \Phi(k), v^*(k')((N_D + 1)^{-\frac{1}{2}} \otimes \mathbb{1}) \Phi(k') \right\rangle d^3k d^3k' \\ &\leq \left[ \int \omega(k)^{-\frac{1}{2}} \|v^*(k)(N_D + 1)^{-\frac{1}{2}}\|_{\mathfrak{F}_D} \|\Phi(k)\| d^3k \right]^2. \end{aligned}$$

Cauchy-Schwarz inequality now implies that

$$\|a(v)u\|^2 \leq \left( \int \omega(k)^{-1} \|v^*(k)(N_D + 1)^{-1/2}\|_{\mathfrak{F}_D}^2 d^3k \right) \left( \int \|\Phi(k)\|^2 d^3k \right),$$

which, together with (52) gives (50).

Let us now prove (51). We have

$$\|a^*(v)u\|^2 = \int \langle v(k) \otimes a^*(k)u, v(k') \otimes a^*(k')u \rangle d^3k d^3k'.$$

Using the commutation relations for  $a$  and  $a^*$  we obtain

$$\begin{aligned} \|a^*(v)u\|^2 &= \int \langle v(k) \otimes a^*(k)a(k')u, v(k') \otimes \mathbb{1}u \rangle d^3k d^3k' \\ &\quad + \int \|(v(k) \otimes \mathbb{1})u\|^2 d^3k \end{aligned} \tag{53}$$

The first term in the right hand side of (53) can be written in the following way

$$\begin{aligned} &\int \langle v(k) \otimes a(k')u, v(k') \otimes a(k)u \rangle d^3k d^3k' \\ &= \int \omega(k)^{-\frac{1}{2}} \omega(k')^{-\frac{1}{2}} \\ &\quad \times \left\langle v(k) ((N_D + 1)^{-\frac{1}{2}} \otimes \mathbb{1}) \Phi(k'), v(k') ((N_D + 1)^{-\frac{1}{2}} \otimes \mathbb{1}) \Phi(k) \right\rangle d^3k d^3k' \end{aligned}$$

Hence

$$\begin{aligned} &\int \langle v(k) \otimes a^*(k)a(k')u, v(k') \otimes \mathbb{1}u \rangle d^3k d^3k' \\ &\leq \left[ \int \omega(k)^{-1/2} \|v(k)(N_D + 1)^{-1/2}\|_{\mathfrak{F}_D}^2 \|\Phi(k)\| d^3k \right]^2 \\ &\leq \left( \int \omega(k)^{-1} \|v(k)(N_D + 1)^{-1/2}\|_{\mathfrak{F}_D}^2 d^3k \right) \left( \int \|\Phi(k)\|^2 d^3k \right) \end{aligned}$$

which, together with (52), yields

$$\begin{aligned} &\int \langle v(k) \otimes a^*(k)a(k')u, v(k') \otimes \mathbb{1}u \rangle d^3k d^3k' \\ &\leq \left( \int \omega(k)^{-1/2} \|v(k)(N_D + 1)^{-1/2}\|_{\mathfrak{F}_D}^2 d^3k \right)^2 \left\| \left( (N_D + 1)^{1/2} \otimes H_{ph}^{1/2} \right) u \right\|^2 \\ &= \left( \int \omega(k)^{-1} \|(N_D + 1)^{-1/2} v(k)^*\|_{\mathfrak{F}_D}^2 d^3k \right) \left\| \left( (N_D + 1)^{1/2} \otimes H_{ph}^{1/2} \right) u \right\|^2. \end{aligned} \tag{54}$$

For the second term in the right hand side of (53), we write

$$\begin{aligned}
& \int \| (v(k) \otimes \mathbb{1}) u \|^2 d^3 k \\
& \leq \left( \int \| v(k) (N_D + 1)^{-\frac{1}{2}} \|^2_{\mathcal{F}_D} d^3 k \right) \| ((N_D + 1)^{\frac{1}{2}} \otimes \mathbb{1}) u \|^2 \\
& = \left( \int \| v(k) (N_D + 1)^{-\frac{1}{2}} \|^2_{\mathfrak{F}_D} d^3 k \right) ((N_D + 1) u, u).
\end{aligned}$$

Using for all  $\varepsilon > 0$ ,  $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ , we get

$$\begin{aligned}
& \int \| (v(k) \otimes \mathbb{1}) u \|^2 d^3 k \\
& \leq \left( \int \| v(k) (N_D + 1)^{-1/2} \|^2 d^3 k \right) \left[ \varepsilon \| (N_D + 1) u \|^2 + \frac{1}{4\varepsilon} \| u \|^2 \right]
\end{aligned} \tag{55}$$

Finally, (53), (54) and (55) give (51).  $\square$

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